Grade 10 Math

Study Guide

Formulas · worked examples · practice problems with the answer key

6 modules · 58 formulas · 88 practice problems

The Exam Room · formula.expert · metric edition 1
September 2026 · Mathieu Leclerc, with Claude

About this study guide

Who it's for. Students working through Grade 10 Math, and the teachers, tutors and parents printing handouts for them. Print it, copy it, staple it — it's free for classroom use, whole or by the page.

How to use it. Each module teaches its formulas in order — equation, picture, and what every symbol means — then closes with a practice set. Work the problems with a pencil; the answer key waits at the back so honest work gets an honest check. The two-page formula sheet up front is the exam-day companion, and every formula carries its number (Math 10-1 onward) so your written work can cite exactly which one you used.

Do it live. Every problem in here has a living version in the Exam Room at **formula.expert/exams/grade-10-math** — new numbers every attempt, hints that pop down when you want them, stars and streaks when you don't need them. When a printed problem stumps you, the matching lesson will coach you through it, step by step, free and with no sign-up. Other courses' study guides live at **formula.expert/study-guides**.

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The practice problems are edition-drawn: a later edition deals fresh numbers, so keep the key with the printing it came from. The Exam Room deals new numbers every attempt.

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The formula sheet

Module 1 · Proportion & Percent

Math 10-1 Unit Price (Price per Unit Quantity)

$$p = \frac{C}{m}$$

Price per unit of quantity, the number that makes two package sizes comparable on a shelf.

Math 10-2 Percent Change

$$c = \frac{x_{\text{new}} - x_{\text{old}}}{x_{\text{old}}}$$

Relative change from an old value to a new one, as a fraction of the old value.

Math 10-3 Recipe Scaling (Ingredient for a New Yield)

$$Q_2 = Q_1 \cdot \frac{S_2}{S_1}$$

Ingredient quantity rescaled from the yield a recipe was written for to the yield you actually want.

Math 10-4 Map Scale to Real Distance

$$d = m\,S$$

Real distance from a map measurement times the scale denominator.

Math 10-5 Discount Price (Percentage Off)

$$S = L\,(1 - d)$$

Price after a percentage is taken off the list price, and the discount implied by any pair of list...

Math 10-6 Sales Tax and Total Price

$$T = P\,(1 + r)$$

Total payable from a pre-tax price and a tax rate, and the reverse: recovering the pre-tax price...

Module 2 · Linear Relations & Rates

Math 10-7 Slope Between Two Points

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Steepness of a line as rise over run between two points.

Math 10-8 Slope-Intercept Form of a Line

$$y = mx + b$$

Equation of a straight line from its slope and y-intercept.

Math 10-9 Slope from Standard Form of a Line

$$m = -\frac{A}{B}$$

Slope of a line written in standard form Ax + By = C, read straight off the two coefficients...

Math 10-10 x-Intercept of a Line

$$x_{\text{int}} = -\frac{b}{m}$$

Where a line in slope-intercept form crosses the x-axis, found by setting y to zero and solving the...

Math 10-11 Point-Slope Form of a Line

$$y = y_1 + m(x - x_1)$$

Equation of a straight line through one known point with a known slope, giving the y-value at any x...

Math 10-12 Speed, Distance & Time

$$v = \tfrac{d}{t}$$

Average speed from distance travelled and elapsed time.

Math 10-13 Fuel Consumption (L/100 km)

$$C = \frac{100\,V}{d}$$

Fuel consumption in litres per hundred kilometres from the fuel burned and the distance covered.

Math 10-14 Miles per Gallon ↔ L/100 km

$$C = \frac{235.215}{M}$$

The reciprocal relation between litres per hundred kilometres and US miles per gallon: their...

Math 10-15 Midpoint Formula

$$x_m = \frac{x_1 + x_2}{2}$$

The coordinate halfway between two endpoints along one axis; apply it to x and again to y for the...

Math 10-16 Distance Formula (2D)

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Straight-line distance between two points in the coordinate plane.

Math 10-17 Pythagorean Theorem

$$a^2 + b^2 = c^2$$

Relates the three sides of a right triangle.

Module 3 · Area & Perimeter

Math 10-18 Rectangle Area

$$A = l \cdot w$$

Area of a rectangle as length times width.

Math 10-19 Area of a Triangle

$$A = \tfrac{1}{2}bh$$

Area from base and perpendicular height.

Math 10-20 Area of a Circle

$$A = \pi r^2$$

Area enclosed by a circle of radius r.

Math 10-21 Trapezoid Area

$$A = \frac{a + b}{2} \cdot h$$

Area of a trapezoid as the average of its two parallel sides times the perpendicular height.

Math 10-22 Parallelogram Area

$$A = b \cdot h$$

Area of a parallelogram from its base and perpendicular height.

Math 10-23 Rectangle Perimeter

$$P = 2(l + w)$$

Perimeter of a rectangle from its length and width.

Math 10-24 Square Perimeter

$$P = 4s$$

Perimeter of a square as four times its side length.

Math 10-25 Parallelogram Perimeter

$$P = 2(a + b)$$

Perimeter of a parallelogram from its two side lengths.

Math 10-26 Triangle Perimeter

$$P = a + b + c$$

Perimeter of a triangle as the sum of its three sides.

Math 10-27 Circumference of a Circle

$$C = 2\pi r$$

Distance around a circle of radius r.

Math 10-28 Ellipse Perimeter (Ramanujan Approximation)

$$P \approx \pi \left[3(a + b) - \sqrt{(3a + b)(a + 3b)} \right]$$

Perimeter of an ellipse by Ramanujan's second approximation — accurate to better than one part in a...

Math 10-29 Circular Sector Area

$$A = \frac{1}{2} r^2 \theta$$

Area of a pie-slice sector of a circle from its radius and central angle in radians.

Math 10-30 Arc Length

$$s = r\theta$$

Length of a circular arc as radius times central angle in radians.

Module 4 · Volume & Surface Area

Math 10-31 Rectangular Prism Volume

$$V = l \cdot w \cdot h$$

Volume of a box as the product of its length, width, and height.

Math 10-32 Cube Volume

$$V = a^3$$

Volume of a cube as its edge length raised to the third power.

Math 10-33 Triangular Prism Volume

$$V = \tfrac{1}{2} b h_t L$$

Volume of a triangular prism — the triangular end area times the prism length, from the triangle's...

Math 10-34 Prism Volume (General Cross-Section)

$$V = BL$$

Volume of any prism or extrusion whose cross-section does not change along its length:...

Math 10-35 Cylinder Volume

$$V = \pi r^2 h$$

Volume of a right circular cylinder from its radius and height, using π ≈ 3.14159265.

Math 10-36 Cone Volume

$$V = \frac{1}{3}\pi r^2 h$$

Volume of a right circular cone — one-third of the matching cylinder — using π ≈ 3.14159265.

Math 10-37 Pyramid Volume

$$V = \frac{1}{3} B h$$

Volume of any pyramid as one-third of its base area times its perpendicular height.

Math 10-38 Sphere Volume

$$V = \frac{4}{3}\pi r^3$$

Volume enclosed by a sphere of radius r, using π ≈ 3.14159265.

Math 10-39 Sphere Surface Area

$$S = 4\pi r^2$$

Surface area of a sphere of radius r, using π ≈ 3.14159265.

Math 10-40 Rectangular Prism Surface Area

$$A = 2(lw + lh + wh)$$

Total surface area of a box — three pairs of matching rectangular faces, from length, width and...

Math 10-41 Cube Surface Area

$$S = 6a^2$$

Surface area of a cube as six times the area of one square face.

Math 10-42 Cylinder Surface Area

$$S = 2\pi r^2 + 2\pi r h$$

Total surface area of a closed cylinder — two end caps plus the wrapped side — using π ≈ 3.14159265.

Math 10-43 Cylinder Lateral Surface Area

$$A = 2\pi r h$$

Area of a cylinder's curved wall only, with no end caps — the label on a can, the insulation around...

Math 10-44 Cone Lateral Surface Area

$$A = \pi r l$$

Curved surface of a right circular cone from its base radius and slant height, excluding the base...

Math 10-45 Cone Slant Height

$$l = \sqrt{r^2 + h^2}$$

Slant height of a right circular cone by Pythagoras on its axial cross-section, from base radius...

Math 10-46 Hemisphere Volume

$$V = \frac{2}{3}\pi r^3$$

Volume of half a sphere, exactly half of the full sphere's (4/3)*pi*r^3.

Math 10-47 Hemisphere Total Surface Area

$$A = 3\pi r^2$$

Total surface of a solid hemisphere: the curved half-sphere 2*pi*r^2 plus the flat circular base...

Module 5 · Right-triangle trigonometry

Math 10-48 Right-Triangle Sine Ratio (SOH)

$$\sin\theta = \frac{o}{h}$$

Relates an acute angle of a right triangle to its opposite side and the hypotenuse.

Math 10-49 Right-Triangle Cosine Ratio (CAH)

$$\cos\theta = \frac{a}{h}$$

Relates an acute angle of a right triangle to its adjacent side and the hypotenuse.

Math 10-50 Right-Triangle Tangent Ratio (TOA)

$$\tan\theta = \frac{o}{a}$$

Relates an acute angle of a right triangle to the ratio of its opposite and adjacent legs.

Math 10-51 Percent Grade from Rise and Run

$$G = \frac{100\,\Delta h}{L}$$

Percent grade is one hundred times the vertical rise divided by the horizontal run, measured level,...

Math 10-52 Grade to Slope Angle

$$\theta = \arctan\left(\frac{G}{100}\right)$$

Converts a percent grade into the slope angle measured from horizontal, and back again with the...

Math 10-53 Slope Ratio (H:V) to Percent Grade

$$G = \frac{100}{n}$$

Converts an embankment slope quoted as n horizontal to one vertical into the equivalent percent...

Module 6 · Money & Interest

Math 10-54 Simple Interest

$$I = P\,r\,t$$

Interest earned on a principal at a flat rate per period, without compounding.

Math 10-55 Compound Interest (Periodic)

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Amount after t periods when interest compounds n times per period at rate r.

Math 10-56 Effective Annual Rate from a Nominal Rate

$$EAR = \left(1 + \frac{r}{m}\right)^m - 1$$

What a quoted nominal annual rate really costs or earns once it compounds m times a year — 12%...

Math 10-57 Rule of 72 (Doubling Time)

$$n \approx \frac{0.72}{i}$$

Mental-arithmetic estimate of how many periods it takes money to double at rate i — with the rate...

Math 10-58 Present Value

$$PV = \frac{FV}{(1+r)^t}$$

Today's value of a future amount, discounted at rate r per period.

MODULE 1

Proportion & Percent

Reading rates · The better buy · Scaling the recipe · Reading the map · Percent off · Adding the tax · Percent change · The Checkout

1.1 Unit Price (Price per Unit Quantity)

(Math 10-1) $$p = \frac{C}{m}$$

Where,

p = Unit price ($/kg)
C = Total price ($)
m = Quantity (kg)

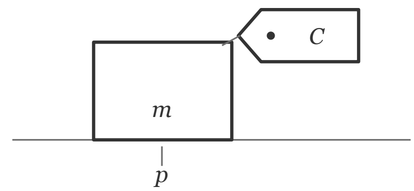

Figure 1.1 — unit price (price per unit quantity).

The only arithmetic that makes two packages comparable: divide the price by the quantity and compare the results. Supermarkets in many jurisdictions are required to print it on the shelf tag precisely because shoppers otherwise cannot do it in their heads, and the required unit is often chosen to be awkward for exactly one of the two products.

The folk rule that the bigger box is cheaper per unit is true often enough to be dangerous. Studies of shelf pricing routinely find a quantity surcharge on the largest size in a fifth or more of product lines, usually where the large size is the one on promotion elsewhere or the one shoppers assume they need not check. The other trap is comparing across different measures: price per kilogram of a drained weight against price per kilogram of a net weight is not a comparison at all, and the same goes for concentrated versus ready-to-use.

Worked example 1.1. Given C = 12.50 $, m = 2.5 kg; find p (unit price). *$12.50 for 2.5 kg → $5.00/kg.*

1.2 Percent Change

(Math 10-2) $$c = \frac{x_{\text{new}} - x_{\text{old}}}{x_{\text{old}}}$$

Where,

c = Relative change
x_{new} = New value
x_{old} = Old value

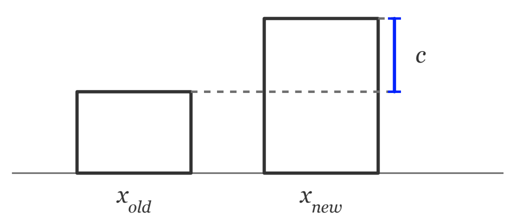

Figure 1.2 — percent change.

Percent change measures a difference relative to where you started: a price moving from $80 to $92 changed by (92 − 80)/80 = 0.15, a 15% rise; a fall from $80 to $68 gives −0.15. Pick the % display unit to see it as a percentage directly, and note that the old value is always the reference — which is why a 50% loss needs a 100% gain to break even.

The rearrangements handle both everyday directions: what a $60 jacket costs after a 30% markup (60 × 1.30 = $78), and the reverse — the pre-sale price of an item now $45 after a 25% discount, 45 / 0.75 = $60, not $56.25 as adding 25% back would wrongly suggest.

Worked example 1.2. Given x_{new} = 100, x_{old} = 80; find c (relative change). *80 → 100 is a +25% change (c = 0.25).*

1.3 Recipe Scaling (Ingredient for a New Yield)

(Math 10-3) $$Q_2 = Q_1 \cdot \frac{S_2}{S_1}$$

Where,
Q_2 = Scaled quantity (kg)
Q_1 = Original quantity (kg)
S_1 = Original servings
S_2 = Desired servings

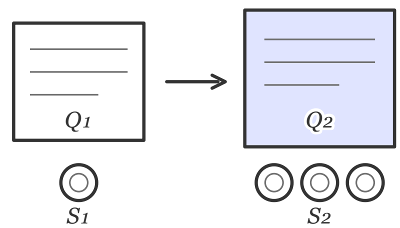

Figure 1.3 — recipe scaling (ingredient for a new yield).

Ingredients scale linearly with servings, so a recipe for four becomes a recipe for six by multiplying every quantity by $6/4$. That much is just proportion, and it is the part a calculator can do for you.

The part it cannot do is everything else in the kitchen. Cooking time does not scale, because heat has to travel into the middle of the food and a doubled loaf is not twice as thick. Pan size scales with area rather than volume, so doubling a cake batter and pouring it into the same tin gives you a raw centre and a burnt rim. Salt, leavening and spice usually want less than a full multiple, since perception is nonlinear and a tripled recipe rarely wants triple the chili. Evaporation scales with surface area too, which is why a doubled sauce reduces more slowly and a doubled pot of rice needs proportionally less water.

Scale the ingredients with the arithmetic, then adjust the seasoning by taste and the timing by testing. Bakers who need real precision go further and work in baker's percentages, where every ingredient is quoted as a fraction of the flour mass and the recipe becomes scale-free by construction.

Worked example 1.3. Given Q_1 = 250 g, S_1 = 4 , S_2 = 6 ; find Q_2 (scaled quantity). *250 g for 4 servings → 375 g for 6.*

1.4 Map Scale to Real Distance

(Math 10-4) $$d = m\,S$$

Where,
d = Real distance (km)
m = Measured map length (cm)

S = Scale denominator

1:50,000 means exactly what it says: one of anything on the map is 50,000 of the same thing on the ground. One centimetre becomes 50,000 cm — 500 m — so 4 cm is 2 km, and the arithmetic never gets harder than that. The scale denominator is a pure number; the units you measure with are the units you get back, multiplied.

The field trick worth owning: a winding trail defeats a ruler, so lay a piece of string along the bends, mark it, pull it straight, and measure THAT. And going the other way — pacing ground and dividing by the scale — turns your boots into a map-measuring tool, which is how a paced 600 m becomes a 1.2 cm check mark on the 1:50,000 sheet.

Worked example 1.4. Given m = 4 cm, S = 50000 ; find d (real distance). *4 cm at 1:50,000 → 2 km on the ground.*

1.5 Discount Price (Percentage Off)

(Math 10-5) $$S = L\,(1 - d)$$

Where,
S = Sale price ($)
L = List price ($)
d = Discount

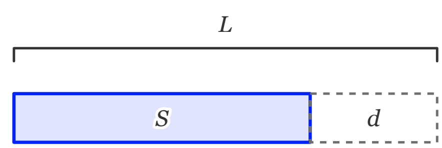

Figure 1.5 — discount price (percentage off).

A percentage off multiplies the list price by what remains: 30% off $250 leaves $250 \times 0.70 = \$175$. Solved the other way, any pair of list and sale prices reveals the discount actually given, which is the useful direction when a supplier quotes you a net price and you want to know where you sit on their schedule.

Two things trip people up. Successive discounts do not add: 20% off followed by a further 10% off is $0.8 \times 0.9 = 0.72$, a 28% discount rather than 30%. And a discount is not the mirror of a markup. Marking $100 up by 50% gives $150, but taking 50% off $150 gives $75, not $100. To undo a markup you divide, and the equivalent discount is always the smaller percentage.

Worked example 1.5. Given L = 250 $, d = 30 %; find S (sale price). *30% off $250 → $175.*

1.6 Sales Tax and Total Price

(Math 10-6) $$T = P\,(1 + r)$$

Where,
T = Total including tax ($)
P = Price before tax ($)
r = Tax rate

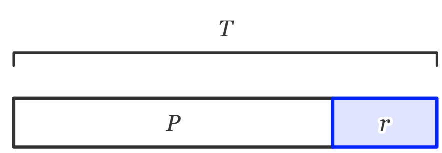

Figure 1.6 — sales tax and total price.

Adding tax is easy: multiply by $1 + r$. A \$100 job at 13% HST totals \$113. Taking tax back *out* of a tax-included figure is where people go wrong, because the instinct is to subtract 13% of the total. Do that to \$113 and you get \$98.31, which is not the price you started from. The correct move is to divide: $113/1.13 = 100$.

The reason is that the tax was calculated on the smaller number, not the larger one. This matters whenever you quote all-in prices to residential customers and then have to report the pre-tax revenue, and it matters at every reconciliation between a cash drawer and a return. The shortcut worth memorising is that backing out a 13% tax means dividing by 1.13, or equivalently multiplying by 0.885.

Worked example 1.6. Given P = 100 \$, r = 13 %; find T (total including tax). *\$100 plus 13% tax → \$113.00 total.*

Module 1 practice

1. The better buy — At the grocery store, a bag of oats holds 5 kg and its tag reads \$25. *Calculate the price per kilogram.*

2. The better buy — Two brands of the same granola sit side by side. Brand A: 3 kg for \$12. Brand B: 4 kg for \$10. *Calculate the price per kilogram of each brand to find the better buy.*

3. Scaling the recipe — A caterer's batch, already scaled up to 6 servings, uses 240 g of pasta. The original recipe made 4 servings. *Determine how much pasta the original recipe called for.*

4. Scaling the recipe — A caterer's batch, already scaled up to 6 servings, uses 240 g of pasta. The original recipe made 4 servings. *Determine how much pasta the original recipe called for.*

5. Reading the map — On a 1:100,000 map, a lakeside trail measures 5 cm with a ruler. *Determine the real distance, in kilometres.*

6. Reading the map — On a 1:50,000 map, the walk to the lookout measures 7 cm with a ruler. *Determine the real distance, in kilometres.*

7. Percent off — In a storewide sale, a pair of runners listed at \$200 is marked 30% off. *Calculate the sale price.*

8. Percent off — A clearance rack marks everything 25% off, and the new tag on a hoodie reads \$120. *Determine the original list price.*

9. Adding the tax — At the till, a pair of headphones rings up at \$60 before tax. Sales tax where you are shopping is 13%. *Calculate the total the register asks for.*

10. Adding the tax — At the till, a pair of headphones rings up at \$120 before tax. Sales tax where you are shopping is 10%. *Calculate the total the register asks for.*

11. Percent change — Last year a video game cost \$40. This year the price is \$50. *Determine the percent change in the price. (A drop is negative.)*

12. Percent change — A ski-hill day pass costs \$90 today. A price cut of 10% is announced for next season. *Calculate next season's price.*

13. The Checkout — Sticker to till, no calculator. A jacket's sticker reads \$40. Today it is 25% off, and tax where you are is 20%. Work each line — every answer feeds the next. *Determine the sale price, the total at the till, and the true percent change from sticker to till.*

14. The Checkout — Bonus mark, still no calculator. On the way out you pass two bags of trail mix. Bag A: 2 kg for \$10. Bag B: 3 kg for \$18. *Determine the better buy's price per kilogram.*

MODULE 2

Linear Relations & Rates

Rise over run · Slope from two points · y = mx + b · Other forms of the line · Speed is a slope · At the pump · Midpoint and distance · The Delivery Run

2.1 Slope Between Two Points

(Math 10-7) $$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Where,
m = Slope
x_1 = First point x-coordinate (m)
y_1 = First point y-coordinate (m)
x_2 = Second point x-coordinate (m)
y_2 = Second point y-coordinate (m)

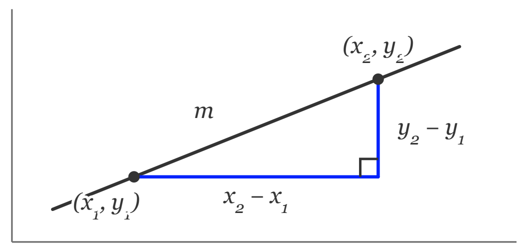

Figure 2.1 — slope between two points.

Slope is rise over run — how far a line climbs for each unit it travels sideways. What makes it a useful number rather than an arbitrary one is that **you get the same answer whichever two points on the line you choose**. That is not a coincidence; it is the defining property of a straight line, and it follows from similar triangles: any two points on the line, dropped to a horizontal and a vertical, form right triangles with the same angles, so their legs stay in the same ratio no matter how far apart you take them. A curve has no slope in this sense, because the ratio would depend on which pair you picked.

A worked instance in units a reader would hold. A wheelchair ramp is limited to a 1:12 slope, or 0.0833. A doorway 400 mm above the walk therefore needs $400/0.0833 = 4.8$ m of run, plus landings — which is why a step that looks trivial so often needs half the front yard. Drainage runs the same arithmetic in the other direction: a sanitary branch falling a quarter inch per foot has a slope of 0.0208, so 6 m of pipe drops 125 mm, and that number decides whether the fixture can reach the stack.

Slope is the ancestor of the derivative. Bring the two points closer and closer together on a curve and the ratio settles on the instantaneous rate of change at that spot, which is the whole of differential calculus in one sentence. It is also worth remembering that a slope carries units in any applied setting — dollars per kilometre, degrees per minute, metres per second. Only on a bare coordinate grid, with both axes in the same units, is it a pure number.

Three things to watch. First, keep the subtractions consistent: $(y_2 - y_1)/(x_2 - x_1)$ is fine and $(y_1 - y_2)/(x_1 - x_2)$ gives the identical answer, because negating both halves leaves the quotient alone — but mixing the two flips the sign. Second, a horizontal line has slope 0 while a vertical line has *no* slope at all. Its run is zero, and the quotient is undefined rather than infinite; these two get swapped constantly. Third, and most common outside the classroom: **a percent grade is not an angle**. Grade is $\tan\theta$ expressed as a percentage, so a 6% grade is 3.4°, not 6°, and a 100% grade is 45° rather than a cliff. Below about 10% the two are close enough that the habit forms; above it, the error grows quickly.

Worked example 2.1. Given x_1 = 1 m, y_1 = 2 m, x_2 = 4 m, y_2 = 8 m; find m (slope). *Points (1,2) and (4,8) → slope 2.*

2.2 Slope-Intercept Form of a Line

(Math 10-8) $y = mx + b$

Where,

y = y-coordinate (m)
m = Slope
x = x-coordinate (m)
b = y-intercept (m)

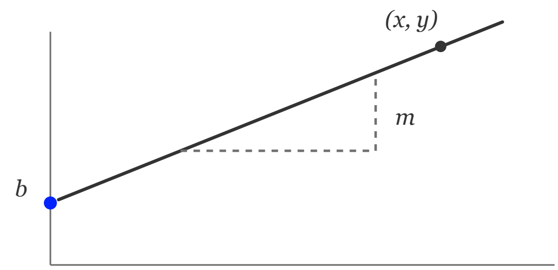

Figure 2.2 — slope-intercept form of a line.

$y = mx + b$ is the workhorse line equation, and it is worth reading as two separate claims. The b is where you start: set $x = 0$ and everything else vanishes, so b is the value of y before anything has happened. The m is the rate at which y changes per unit of x. Between them those two numbers pin down every non-vertical straight line there is, which is why two parameters is all a linear model ever needs.

The reason this outlives school algebra is that it is the shape of any process with a fixed part and a per-unit part. A service call billed at $95 to show up plus $85 an hour is $y = 85x + 95$; four hours on site invoices at $435, and running it backwards, a $520 invoice accounts for 5 hours. Utility bills, freight rates, and equipment rentals are all this equation with different names on the constants, and the useful habit is asking, of any quoted price, which number is the b and which is the m.

Two related forms cover what this one cannot. Point-slope, $y - y_1 = m(x - x_1)$, is the natural choice when you know a point on the line that is not the intercept — a calibration reading, say — and want the line through it. Standard form, $Ax + By = C$, is more general still, and unlike slope-intercept it can describe a vertical line, which $y = mx + b$ is structurally incapable of representing because a vertical line has no slope to put in m.

Four things go wrong. The first is confusing b with the x-intercept: b is where the line crosses the *vertical* axis, while the crossing of the horizontal axis is at $x = -b/m$, which is a different number and often the one actually wanted — a break-even point, for instance. The second is that solving for m divides by x, so a point with $x = 0$ tells you nothing about the slope; every line through the intercept fits it equally. The third is sign carelessness with a negative intercept: $y = 3x - 4$ has $b = -4$, and typing 4 shifts the whole line eight units. The fourth is the conceptual one: **a straight line is a model, not a law**. Fitting one to two measurements and extrapolating far past them assumes a constancy that most real processes do not have, and the intercept in particular frequently describes a condition that never occurs — the value at zero hours, zero flow, or zero temperature may be pure arithmetic with no physical meaning at all.

Worked example 2.2. Given m = 2, x = 3 m, b = 1 m; find y (y-coordinate). *y = 2·3 + 1 → 7 m.*

2.3 Slope from Standard Form of a Line

(Math 10-9) $$m = -\frac{A}{B}$$

Where,
m = Slope
A = Coefficient of x
B = Coefficient of y

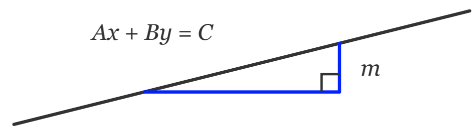

Figure 2.3 — slope from standard form of a line.

Solving Ax + By = C for y gives y = (−A/B)x + C/B, so the slope is −A/B and the y-intercept is C/B — you never need to do the rearrangement in full. Worked example: 6x + 3y = 12 has slope **−2** and intercept 4, matching y = −2x + 4. Notice that C plays no part in the slope at all: 6x + 3y = 12 and 6x + 3y = −100 are parallel lines, differing only in where they sit.

That observation makes standard form the quickest way to spot parallel and perpendicular pairs: two lines are parallel when their A:B ratios match, and perpendicular when the coefficients swap with one sign flipped, so 6x + 3y = 12 is perpendicular to 3x − 6y = 5. Standard form is also the shape linear programming and systems of equations want, which is why textbooks insist on integer coefficients with A positive. The trap is the minus sign — students routinely read the slope of 6x + 3y = 12 as 2 or as 6/3 rather than −2 — and the vertical case, where B = 0 leaves x = C/A with no slope to report.

Worked example 2.3. Given A = 6, B = 3; find m (slope). *6x + 3y = 12 → slope −2.*

2.4 x-Intercept of a Line

(Math 10-10) $$x_{\text{int}} = -\frac{b}{m}$$

Where,
x_{int} = x-intercept (m)
m = Slope
b = y-intercept (m)

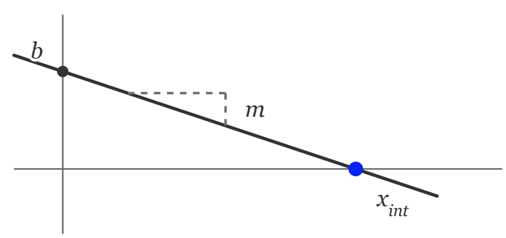

Figure 2.4 — x-intercept of a line.

A line crosses the x-axis where y = 0, so setting 0 = mx + b and solving gives x = −b/m. Worked example: y = 2x − 6 meets the axis at x = 6/2 = **3**, the point (3, 0). It is the same thing as the root of the linear function, and the linear cousin of the quadratic formula — one root instead of two, because a straight line can only cross once.

Intercepts are how you sketch a line in seconds: plot (0, b) and (−b/m, 0), then join them. They also carry the meaning in applied problems. A tank draining as V = 500 − 25t empties when 25t = 500, at t = 20 minutes; a business whose profit runs P = 40n − 1200 breaks even at n = 30 units. The traps are the minus sign (the intercept of y = 2x − 6 is +3, because −(−6)/2 = 3) and the horizontal case, where m = 0 means the line never crosses at all unless it already lies on the axis. Note that b and the x-intercept are both lengths here while the slope is a pure ratio, so you can enter the intercept in one unit and read the crossing point in another.

Worked example 2.4. Given m = 2, b = -6 m; find x_{int} (x-intercept). *y = 2x − 6 → x-intercept 3 m.*

2.5 Point-Slope Form of a Line

(Math 10-11) $$y = y_1 + m(x - x_1)$$

Where,

y = y-coordinate (m)

y_1 = Known point y-coordinate (m)

m = Slope

x = x-coordinate (m)

x_1 = Known point x-coordinate (m)

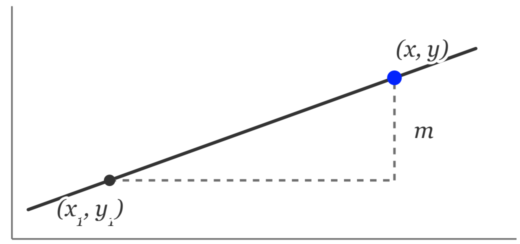

Figure 2.5 — point-slope form of a line.

Point-slope form is the natural way to write a line when you know one point on it and how steep it is, which in practice is nearly always. It is the slope definition m = (y − y_1)/(x − x_1) with the denominator multiplied across, then solved for y so the calculator can evaluate it. Worked example: the line through (3, 4) with slope −2 reaches y = 4 + (−2)(7 − 3) = **−4** at x = 7.

Compared with slope-intercept form, point-slope needs no detour through the y-intercept — handy when the intercept is far off the page or physically meaningless, as with a thermometer calibration that passes through (100 °C, 212 °F) with slope 1.8. Two traps recur. The subtraction is x − x_1, so a known point at x_1 = −5 gives (x + 5); and the point you substitute must actually lie on the line, or every value that follows is wrong. Tangent lines in calculus are written in exactly this form for the same reason: differentiation hands you a point and a slope, never an intercept.

Worked example 2.5. Given y_1 = 4 m, m = -2, x = 7 m, x_1 = 3 m; find y (y-coordinate). *Through (3, 4) with m = −2, at x = 7 → y = −4 m.*

2.6 Speed, Distance & Time

(Math 10-12) $$v = \frac{d}{t}$$

Where,

v = Speed (m/s)

d = Distance (m)

t = Time (s)

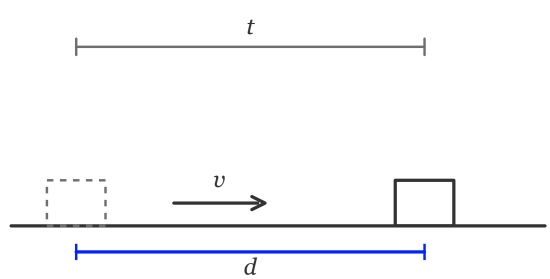

Figure 2.6 — speed, distance & time.

Speed is ground covered divided by time taken, $v = d/t$, and it is worth being clear about what kind of statement that is. It is not a law of nature that could turn out to be false — it is the *definition* of average speed. Nothing in it can be wrong; it can only be misapplied. What it gives you is the single steady speed that would have covered the same distance in the same time, which is a genuinely useful summary of a trip and tells you almost nothing about any particular moment within it.

Take a drive of 148 km that takes 1 hour 45 minutes. Convert the time to a single unit first — 1.75 h — and $v = 148/1.75 = 84.6$ km/h. In SI the same trip is 148 000 m over 6300 s, giving 23.5 m/s. Both answers describe the same drive, and the traffic light you sat at for ninety seconds is buried inside both of them.

This is the zeroth member of the kinematics family: set $a = 0$ in $d = v_0 t + \frac{1}{2}at^2$ and you are left with $d = vt$. Run in the other direction, toward instantaneous speed, it becomes the derivative $v = \mathrm{d}d/\mathrm{d}t$, which is what your speedometer reads and what calculus was partly invented to handle. The relation also underwrites the modern definition of length itself: since 1983 the metre has been defined by *fixing* the speed of light at 299 792 458 m/s exactly, so distance is now measured by timing light and rearranging this formula for d.

The classic mistake is averaging the speeds instead of the trip. Drive 60 km out at 60 km/h and return at 30 km/h, and the average for the round trip is not 45 km/h. The outbound leg takes 1 h, the return 2 h, so it is 120 km in 3 h — 40 km/h. Average speed is always total distance over total time, never the mean of the individual speeds, because you spend longer at the slow one. The second trap is units: this formula does not convert anything, so 100 km and 30 minutes gives 3.33 in units of km/min, not km/h. Fix the units before the arithmetic, or use the unit selectors on this page. And note that this is *distance*, the ground covered, not displacement — a runner who finishes a 400 m lap where she started has run at a respectable speed and has an average velocity of exactly zero.

Worked example 2.6. Given d = 100 m, t = 8 s; find v (speed). *100 m in 8 s → 12.5 m/s.*

2.7 Fuel Consumption (L/100 km)

(Math 10-13) $$C = \frac{100\,V}{d}$$

Where,
C = Consumption (L/100 km)
V = Fuel used (L)
d = Distance (km)

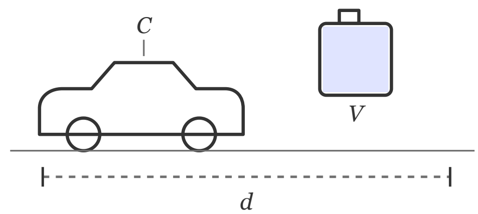

Figure 2.7 — fuel consumption (l/100 km).

Most of the world measures fuel the sensible way round: litres consumed per hundred kilometres driven, so a smaller number is better and the quantity is directly proportional to what you pay. Divide the fuel you put in by the distance you covered since the last fill, scale to 100 km, and you have a figure you can compare against the sticker.

The inverse convention used in the United States creates a genuine cognitive trap known as the MPG illusion, documented by Richard Larrick and Jack Soll in *Science* in 2008. Improving a vehicle from 10 to 15 mpg saves more fuel over the same distance than improving one from 30 to 40 mpg, which almost nobody believes on first hearing. In litres per hundred kilometres the same comparison is obvious at a glance: 23.5 falls to 15.7, a saving of 7.8, while 7.8 falls to 5.9, a saving of 1.9. The metric unit makes the arithmetic honest.

Worked example 2.7. Given V = 45 L, d = 600 km; find C (consumption). *45 L over 600 km → 7.5 L/100 km.*

2.8 Miles per Gallon ↔ L/100 km

(Math 10-14) $$C = \frac{235.215}{M}$$

Where,
C = Consumption (L/100 km)
M = Fuel economy (mpg)

The two conventions are reciprocals with a conversion constant baked in, and the constant is exact: one hundred kilometres is 62.137 miles, and a US gallon is 3.785411784 litres, so $100 \times 3.785411784/1.609344 = 235.2146$. Multiply litres per hundred kilometres by miles per gallon and you always get that number.

Watch which gallon is meant. The imperial gallon used in the UK is 4.546 litres, about 20 % larger, so the same car rates 282.5 divided by its consumption in "imperial mpg" and looks more efficient than it is. This page uses the US gallon, the one on every American window sticker. Because the relationship is reciprocal rather than proportional, equal steps in one unit are not equal steps in the other, which is the whole point of the previous page's warning.

Worked example 2.8. Given M = 30; find C (consumption). *30 mpg → 7.8405 L/100 km.*

2.9 Midpoint Formula

(Math 10-15) $$x_m = \frac{x_1 + x_2}{2}$$

Where,
x_m = Midpoint coordinate (m)
x_1 = First endpoint coordinate (m)
x_2 = Second endpoint coordinate (m)

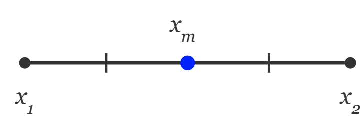

Figure 2.9 — midpoint formula.

The midpoint of a segment is found one axis at a time: average the two x-coordinates for the midpoint's x, then average the two y-coordinates for its y. Worked example: the midpoint of (3, 4) and (11, −2) has x = (3 + 11)/2 = **7** and y = (4 + (−2))/2 = 1, giving (7, 1). Run this calculator twice, once per axis, and you have the point.

The rearrangements answer the question students actually get asked more often: "one end of a segment is at 2 and the midpoint is at 5 — where is the other end?" The answer is x_2 = 2(5) − 2 = 8, not 5 + 2. Doubling the midpoint is the whole trick, and forgetting to double is the single commonest error on this topic. The idea is as old as coordinates themselves — Descartes and Fermat, working independently in the 1630s, turned geometry into arithmetic precisely so that questions like this could be answered by averaging instead of by compass construction. Because both endpoints and the midpoint are lengths, you can mix units freely here: an endpoint in inches and a midpoint in feet convert before the arithmetic runs.

Worked example 2.9. Given x_1 = 3 m, x_2 = 11 m; find x_m (midpoint coordinate). *Midpoint of 3 m and 11 m → 7 m.*

2.10 Distance Formula (2D)

(Math 10-16) $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Where,

x_1 = First point x-coordinate (m)
y_1 = First point y-coordinate (m)
x_2 = Second point x-coordinate (m)
y_2 = Second point y-coordinate (m)
d = Distance between the points (m)

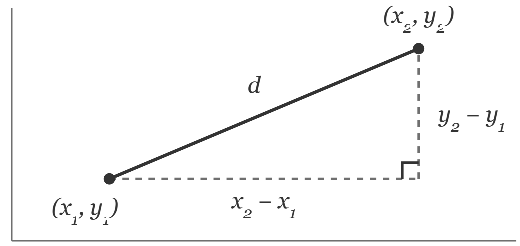

Figure 2.10 — distance formula (2d).

The distance formula is the Pythagorean theorem wearing coordinate clothing. Between any two points, the horizontal separation ($x_2 - x_1$) and the vertical separation ($y_2 - y_1$) form the two legs of a right triangle, and the straight-line distance is its hypotenuse. From (1, 2) to (4, 6) the legs are 3 and 4, so the distance is √(9 + 16) = 5 — the beloved 3–4–5 triangle hiding in the grid.

Historically this formula marks the marriage of algebra and geometry: René Descartes's *La Géométrie* (1637) introduced the coordinate plane that lets shapes be handled as equations, and measuring distance was the first payoff. Today the same square-root-of-summed-squares expression computes GPS displacements, collision distances in games, and error magnitudes in statistics. Note that only d can be solved for directly — recovering a single coordinate from a known distance gives two equally valid mirror-image answers, so there is no unique inverse.

Worked example 2.10. Given x_1 = 1 m, y_1 = 2 m, x_2 = 4 m, y_2 = 6 m; find d (distance between the points). *(1,2) to (4,6) m → 3-4-5 hypotenuse, 5 m.*

2.11 Pythagorean Theorem

(Math 10-17) $a^2 + b^2 = c^2$

Where,

a = Leg a (m)
b = Leg b (m)
c = Hypotenuse c (m)

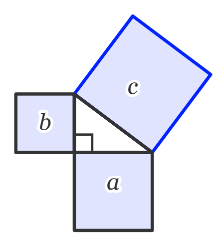

Figure 2.11 — pythagorean theorem.

In a right triangle, $a^2 + b^2 = c^2$, where c is the hypotenuse — the side facing the right angle. Written out, it says that the square built on the long side has exactly the same area as the two squares built on the short sides put together, and that is worth pausing on, because it is a statement about **areas** that lets you calculate a **length**. That is the whole trick, and it is why the squares and the square root are there rather than something simpler.

There are hundreds of proofs; the one worth carrying around takes two copies of a square of side $a + b$. In the first, arrange four copies of the triangle in the corners so the leftover space forms two squares, of areas a^2 and b^2. In the second, slide the same four triangles into a pinwheel so the leftover space is a single tilted square of area c^2. Same big square, same four triangles removed, so what remains must match. No algebra, and you can do it with paper. The converse is true as well, and it is the half that earns its living on site: if three measured lengths satisfy the relation, the angle between the two short ones *is* square. That is why a layout crew measures 3, 4 and 5 to set a corner rather than trusting a framing square.

Use the biggest triangle the space allows — a 3–4–5 in feet leaves a corner good to perhaps half a degree, while a 12–16–20 divides that error by four. A rafter run of 5.4 m with a rise of 2.2 m needs a length of $\sqrt{5.4^2 + 2.2^2} = 5.83$ m before the tail cut. Two triples are worth memorising because they come out whole: 3–4–5 and 5–12–13.

Three ways it goes wrong. The first is putting a leg where the hypotenuse belongs: c is always the longest side, so if the answer comes back shorter than something you typed, the sides are in the wrong slots. The second is dropping the squares — the legs 3 and 4 do not make a 7, they make a 5, and the shortcut across a rectangular lot saves far less than people expect. The third is applying it to a triangle that has no right angle at all; for those, the law of cosines carries a correction term and reduces to this the moment the angle reaches 90°. Solving for a leg has a built-in honesty check: $a = \sqrt{c^2 - b^2}$ needs $c > b$, and if it is not, the square root turns imaginary because the triangle you described cannot be drawn.

Worked example 2.11. Given a = 3 m, b = 4 m; find c (hypotenuse c). *legs 3 m and 4 m → hypotenuse 5 m.*

Module 2 practice

15. Rise over run — A wheelchair ramp outside the library rises 9 m over a horizontal run of 3 m. *Determine the slope of the ramp.*

16. Rise over run — From the top of the sledding hill, the ground drops 3 m over a horizontal run of 6 m. *Determine the slope of the hill.*

17. Slope from two points — On the grid, a line passes through the points P(1, 3) and Q(4, 9). *Calculate the slope of the line through P and Q.*

18. Slope from two points — On the grid, a line passes through the points P(3, 12) and Q(6, 6). *Calculate the slope of the line through P and Q.*

19. y = mx + b — A phone plan charges a flat $5 a month plus $2 for every gigabyte of data. This month's usage was 4 GB. *Calculate this month's bill.*

20. y = mx + b — A ride-share trip of 4 km came to $14, which includes the app's flat $2 booking fee. *Determine the per-kilometre rate — the slope of the fare line.*

21. Other forms of the line — The last question on the homework hands you a line in standard form: 2x + 8y = 16. *Determine the slope of the line.*

22. Other forms of the line — A birthday candle is 8 cm tall when lit and burns down 2 cm every hour, tracing the line y = −2x + 8 on a height–time graph. *Determine when the candle burns out — the x-intercept of its line.*

23. Speed is a slope — A family drives 240 km to the cottage at a steady 80 km/h. *Determine how long the drive takes.*

24. Speed is a slope — A school bus rolls along the highway at a steady 40 km/h for 2 hours. *Determine how far the bus travels.*

25. At the pump — On the drive to visit cousins, the car burns 36 L of gas over 300 km. *Determine the car's fuel consumption, in L/100 km.*

26. At the pump — The pump clicks off at 60 L, priced at $1.50 per litre. *Calculate the cost of the fill-up.*

27. Midpoint and distance — On the schoolyard map, marked in metres, a water fountain will go exactly halfway between the maple at (4, 2) and the oak at (10, 8). *Determine the coordinates of the fountain — the midpoint of the two trees.*

28. Midpoint and distance — On the same schoolyard map, a straight path will run from the gate at (1, 1) to the slide at (9, 7), coordinates in metres. *Calculate the length of the path.*

29. The Delivery Run — Saturday route, numbers picked to fit in your head. The bakery van drives two legs: 120 km of highway at 60 km/h, then 30 km of county road at 30 km/h. It drinks 6 L/100 km the whole way. Work each line — every answer feeds the next. *Determine each leg's time, the trip's average speed, and the fuel it burns — one line at a time.*

30. The Delivery Run — Extra credit, read off the dashboard: over a 150 km route, the trip computer logged 12 L of fuel. *Determine the van's consumption rating.*

MODULE 3

Area & Perimeter

Naming the formula · Rectangles and parallelograms · Triangles and trapezoids · Circles · Sectors and arcs · Area in reverse · Composite floors · The Paint Estimate

3.1 Rectangle Area

(Math 10-18) $A = l \cdot w$

Where,
A = Area (m²)
l = Length (m)
w = Width (m)

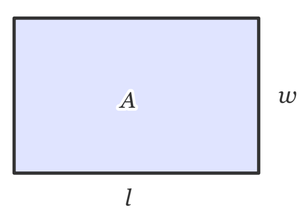

Figure 3.1 — rectangle area.

Length times width is the most fundamental area formula there is: a rectangle 4 units long and 3 units wide tiles perfectly into 4 × 3 = 12 unit squares, and counting those squares *is* the area. A painter uses it constantly — a wall 4 m long and 2.5 m high presents 10 m² of surface, and at a typical coverage of 10 m² per litre, that wall costs exactly one litre of paint per coat.

Every other area formula is built on this one. A parallelogram is a sheared rectangle, a triangle is half of one, and even the areas of curved shapes are found by slicing them into ever-thinner rectangular strips — the idea at the heart of integral calculus. Watch the units: multiplying two lengths gives a squared unit, so metres in means square metres out, and mixing feet with metres without converting first is the classic mistake.

Worked example 3.1. Given l = 8 m, w = 5 m; find A (area). *8 m x 5 m rectangle → 40 m².*

3.2 Area of a Triangle

(Math 10-19) $A = \frac{1}{2}bh$

Where,

A = Area (m²)

b = Base (m)

h = Height (m)

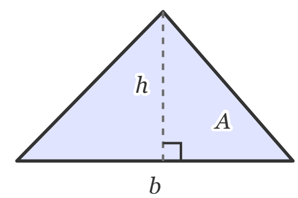

Figure 3.2 — area of a triangle.

$A = \frac{1}{2}bh$, and the proof takes one move: make a second copy of your triangle, turn it half a turn, and slide it against the first. The two fit together into a parallelogram of base b and height h, with no gap and no overlap, so one triangle is half of it. A parallelogram is in turn a sheared rectangle — cut the triangle off one end and it fits exactly onto the other — so every triangle traces back to length times width, the definition of area itself. That is the entire content of the formula, and it is why a triangle is the atom of area measurement: any polygon whatever can be cut into triangles, and that is precisely how surveying software computes the area of an irregular lot.

A consequence worth having: since only the perpendicular height enters, sliding the apex sideways along a line parallel to the base changes the shape but not the area. A tall thin sliver and a tidy isosceles triangle on the same base, with the same height, cover exactly the same ground.

A worked instance. The gable end of a garage is 6.4 m wide at the plate and rises 2.1 m to the ridge, so $A = \frac{1}{2}(6.4)(2.1) = 6.72$ m² — about two and a half sheets of drywall, or one litre of paint per coat at typical coverage. Backwards, $h = 2A/b$ and $b = 2A/h$: a triangular flowerbed of 5 m² against a 4 m wall must run $2(5)/4 = 2.5$ m out from it.

Where it goes wrong is almost always the height. h is the **perpendicular** distance from the base to the opposite vertex, not the length of the slanted side that runs up to it — and on a drawing the slant is the number that is written down, because it is the one somebody could measure. Using it inflates the answer, badly on a steep triangle. Two more. On an obtuse triangle the foot of that perpendicular lands *outside* the base, so the height has to be measured to an extension of the base line; the formula still holds, but the picture stops being reassuring. And b and h must be a matched pair — any of the three sides can serve as the base, but the height must be the one dropped to *that* side. If all you have is the three side lengths, this formula cannot help you and Heron's can; if you have two sides and the angle between them, use $A = \frac{1}{2}ab\sin C$, which is this same formula with $b \sin C$ quietly supplying the height.

Worked example 3.2. Given b = 10 m, h = 6 m; find A (area). *base 10 m, height 6 m → 30 m².*

3.3 Area of a Circle

(Math 10-20) $A = \pi r^2$

Where,

A = Area (m²)

r = Radius (m)

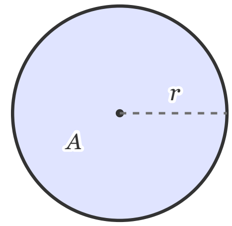

Figure 3.3 — area of a circle.

$A = \pi r^2$. Most people meet this as something to memorise, so here is why it is true, in a picture you can cut out of paper. Slice the disc into a great many thin concentric rings and cut each ring open, then lay them in a stack, longest at the bottom. The outermost ring straightens into a strip of length $2\pi r$; the innermost is almost nothing; the ones between shorten evenly. The stack is a triangle of base $2\pi r$ and height r, and half base times height gives $\frac{1}{2}(2\pi r)(r) = \pi r^2$. The circle's area is not a separate fact from its circumference — it is the circumference integrated outward from the centre, which is also why the derivative of πr^2 is $2\pi r$.

A worked instance in units you would hold. A circular concrete pad 3.0 m across has $r = 1.5$ m, so $A = \pi(1.5)^2 = 7.07$ m². At 100 mm thick that is 0.71 m³ of concrete, and you would order 0.8 to cover the spill and the screed. Backwards, $r = \sqrt{A/\pi}$ answers the other half of the same job: 12 m² of pad wants a circle 3.91 m in radius, so 7.8 m across.

The squared radius is the part that surprises people, and it answers more real questions than the formula does. Area scales as the square of size, so doubling the diameter **quadruples** the area. A 2-inch pipe carries four times the cross-section of a 1-inch pipe. A sprinkler that throws twice as far waters four times the lawn. And two 9-inch pizzas (127 in² between them) beat one 12-inch (113 in²), which is the version of this fact most likely to come up over dinner. It also explains why upsizing anything round is rarely a small decision.

Now the mistake, and it is the commonest wrong answer in all of geometry: **entering the diameter where the radius belongs**. Nobody measures a radius. You measure across the thing, at the widest point if you are lucky, and halve it. Forget the halving and the squaring multiplies your error by four — a tank you thought held 700 L holds 175. Keep one sanity check in your pocket: a circle always fills a bit under 80% of the square that boxes it, because $\pi/4 = 0.785$. So a circle 3 m across sits inside a 9 m² square and must come out around 7 m². If your answer is 28, you used the diameter. The other trap is units — squaring a length squares its unit, so metres in means square metres out, and a radius in centimetres with an area wanted in square metres needs converting first, not at the end.

Worked example 3.3. Given r = 1 m; find A (area). *r = 1 m → A = pi = 3.14159 m².*

3.4 Trapezoid Area

(Math 10-21) $$A = \frac{a+b}{2} \cdot h$$

Where,

A = Area (m²)

a = Parallel side a (m)

b = Parallel side b (m)

h = Height (m)

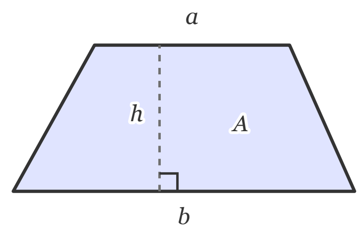

Figure 3.4 — trapezoid area.

A trapezoid's area is the average of its two parallel sides multiplied by the perpendicular distance between them. The proof fits in one picture: take a second, identical trapezoid, flip it upside down, and snug it against the first — together they form a parallelogram with base (a + b) and height h, so one trapezoid is half of that. Area rules of exactly this kind appear in the Rhind papyrus, where Egyptian scribes around 1550 BC computed the areas of tapering fields along the Nile.

The formula still earns its keep in civil engineering: a drainage channel with a 3 m bottom, a 5 m top, and a 1.2 m depth has a cross-section of (3 + 5)/2 × 1.2 = 4.8 m², the number that sets how much water it can carry. As always, h must be measured perpendicular to the parallel sides — never along a slanted leg.

Worked example 3.4. Given a = 3 m, b = 5 m, h = 4 m; find A (area). *sides 3 m and 5 m, height 4 m → 16 m².*

3.5 Parallelogram Area

(Math 10-22) $$A = b \cdot h$$

Where,

A = Area (m²)

b = Base (m)

h = Height (m)

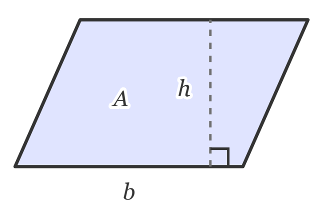

Figure 3.5 — parallelogram area.

A parallelogram is a rectangle that has been pushed sideways, and the proof that its area is still base times height is a single cut. Slice the triangle off the leaning end, carry it round to the other end, and it fits exactly — what you are holding is now a rectangle with the same base and the same height. Nothing was added and nothing was lost, so $A = bh$, the rectangle's own formula, unchanged. The more general statement is Cavalieri's principle: two shapes sliced at every height into strips of equal length must have equal area, and shearing slides the strips sideways without changing a single one of their lengths.

A worked instance. Surveyed lots are often skewed to follow a road, and the arithmetic does not care. A parcel with 24 m of frontage and a depth of 31 m measured square to that frontage covers $24 \times 31 = 744$ m² whether the side lines run perpendicular or lean at 20°. Solving backwards, $h = A/b$ and $b = A/h$: 744 m² fronting 31 m must be 24 m deep.

Shear invariance turns out to be one of the load-bearing ideas in mathematics rather than a curiosity about quadrilaterals. It is why the determinant measures area —

the parallelogram spanned by two vectors has area $|x_1y_2 - x_2y_1|$, and row operations that shear a matrix leave the determinant alone for exactly this reason. It is also the two-dimensional version of the argument that gives a pyramid one third of its prism.

The dominant error is using the **slanted side** where the perpendicular height belongs. The two are not close: they differ by a factor of $\sin\theta$, so a parallelogram leaning 30° off square has a height only 87% of its side, and the area comes out 15% high. The slant is the number printed on drawings and the only one a tape can reach along an edge, which is why it keeps getting typed in. If the side s and the lean angle are what you have, use $A = bs\sin\theta$ instead, or convert first with $h = s\sin\theta$. Two smaller traps: on a sharply leaned parallelogram the foot of the height lands outside the base, so the perpendicular has to be dropped to an extension of the base line and the picture stops looking like the formula; and the side lengths alone will never give you the area, because a hinged parallelogram sweeps from a full rectangle down to a flat line with every side length unchanged the whole way.

Worked example 3.5. Given b = 12 m, h = 7 m; find A (area). *base 12 m, height 7 m → 84 m².*

3.6 Rectangle Perimeter

(Math 10-23) $$P = 2(l + w)$$

Where,
P = Perimeter (m)
l = Length (m)
w = Width (m)

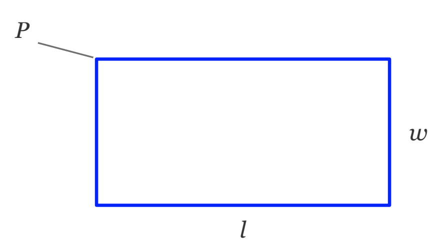

Figure 3.6 — rectangle perimeter.

Two lengths and two widths make the trip around, so $P = 2(l + w)$. It is the baseboard formula, the fence formula, the picture-frame formula: a 4 m × 3 m room needs 14 m of trim, minus the doorways. Solving backwards is just as common — with 30 ft of edging and a 10 ft side already fixed, the other side must be 5 ft.

The contrast with area is worth holding onto. Perimeter grows in step with size while area grows with its square, so of all rectangles with a 20 m perimeter, the 5 × 5 square encloses 25 m² while a long thin 9 × 1 encloses only 9 m². Same fencing, very different paddock.

Worked example 3.6. Given l = 8 m, w = 5 m; find P (perimeter). *8 m x 5 m → perimeter 26 m.*

3.7 Square Perimeter

(Math 10-24) $P = 4s$

Where,
P = Perimeter (m)
s = Side length (m)

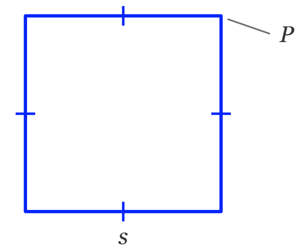

Figure 3.7 — square perimeter.

Four equal sides, so the distance around is simply four times one of them. The formula earns its keep in the fencing aisle: 100 ft of fencing encloses a square 25 ft on a side, covering 625 ft² — and no other rectangle made from that same 100 ft does better. That is the isoperimetric principle in miniature: among all rectangles of a given perimeter, the square holds the most area, and among all shapes whatsoever, the circle does.

Comparing $P = 4s$ with $A = s^2$ also shows why big things are cheaper to wrap than small ones. Double the side and the perimeter doubles, but the area quadruples — the reason bulk packaging uses less cardboard per litre, and the reason a small animal loses body heat faster than a large one.

Worked example 3.7. Given s = 5 m; find P (perimeter). *5 m square → perimeter 20 m.*

3.8 Parallelogram Perimeter

(Math 10-25) $P = 2(a + b)$

Where,
P = Perimeter (m)
a = Side a (m)
b = Side b (m)

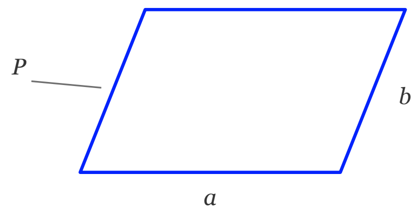

Figure 3.8 — parallelogram perimeter.

$P = 2(a + b)$, the same structure as a rectangle's perimeter, and for a reason worth knowing. A parallelogram is *defined* by having both pairs of opposite sides parallel — nothing is said about their lengths. That they come out equal is a theorem: draw either diagonal, and the parallel lines make alternate angles match, so the two triangles are congruent and their corresponding sides agree. The formula is the payoff of that proof, not a definition.

The property that makes this shape interesting is that shearing it — leaning it over — changes the area but never the perimeter. Hinge four rods into a parallelogram and push: the sides keep their lengths the whole way, so P never moves, while the perpendicular height collapses and the area falls to zero. That is not an abstraction. It is exactly why a garden gate sags into a rhombus over a winter, and why the cure is a diagonal brace running from the bottom hinge upward — a triangle cannot be sheared, because changing its shape would mean changing a side length. Scissor lifts, lazy-tong extensions and parallel-motion drafting arms use the same collapsibility on purpose.

A worked instance: a parallelogram-shaped raised bed with sides of 3.2 m and 1.8 m needs $2(3.2+1.8)=10$ m of edging, regardless of how sharply it is skewed to fit the fence line. Backwards, $a=P/2-b$: 10 m of edging with one side fixed at 3.2 m leaves 1.8 m for the other. The solver refuses cases where half the perimeter does not exceed the known side, since the remaining side would come out zero or negative.

Two errors recur. The first is feeding in the **perpendicular height** instead of the slanted side. The height is the number printed on most drawings because it is what the area formula wants, and it is always shorter than the side it stands in for — on a steeply leaned shape, considerably shorter, so the perimeter comes out low and you order too little edging. The two coincide only when the parallelogram is a rectangle. The second is expecting the perimeter to say something about the area. It does not: a 3.2 × 1.8 rectangle encloses 5.76 m², while the same four sides leaned to 30° enclose 2.88 m². Of all parallelograms with given sides, the rectangle holds the most.

Worked example 3.8. Given a = 5 m, b = 3 m; find P (perimeter). *sides 5 m and 3 m → perimeter 16 m.*

3.9 Triangle Perimeter

(Math 10-26) $$P=a+b+c$$

Where,
P = Perimeter (m)
a = Side a (m)
b = Side b (m)
c = Side c (m)

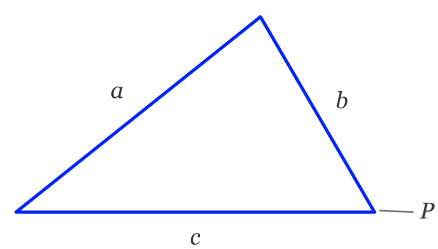

Figure 3.9 — triangle perimeter.

Adding three numbers is not where the interest lies. The real content of this page is which triples of numbers are **allowed**, and the rule is the triangle inequality: each side must be shorter than the other two put together. Lengths of 2, 3 and 10 will not close into a triangle no matter how you arrange them, and the reason is not a rule someone imposed — it is that the straight run from one corner to another is the shortest path between those two points, so detouring through a third corner can never be shorter. If $a+b$ exactly equalled c, the triangle would flatten into a line segment with no area at all, which is the boundary case the solver rejects.

The usual working direction is backwards. A surveyor closing a traverse, or anyone with a perimeter and two known sides, gets the third by subtraction: a triangular lot with 78 m of frontage in total, with two sides of 30 m and 26 m, has a third side of 22 m. That answer is legal — 22 + 26 exceeds 30 — but note that arithmetic alone would not have told you so, and a perimeter of 78 with sides of 30 and 46 gives a third side of 2 m that cannot be built.

Half the perimeter has a name and a life of its own. The semiperimeter $s=P/2$ is the gateway to Heron's formula, which extracts the area from the three sides alone, and to the incircle, whose radius is simply $r=A/s$. Both are reasons to compute a perimeter you did not think you needed. The triangle inequality itself outgrew geometry entirely: it is one of the three defining axioms of a metric space, so every notion of distance in mathematics — between functions, between signals, between DNA sequences — is required to obey the same rule these three fence lines do.

Two things to watch. Perimeter scales *linearly* with size while area scales as the square, so a lot with twice the dimensions needs twice the fencing but encloses four times the ground — fencing is cheap per hectare on large parcels and expensive on small ones. And the perimeter alone never determines the shape: an equilateral triangle of perimeter 78 m encloses 293 m², while a 30–26–22 triangle of the same perimeter encloses 285 m², and a nearly flat 38–38–2 encloses only 38 m². Same fence, very different paddock.

Worked example 3.9. Given a = 3 m, b = 4 m, c = 5 m; find P (perimeter). *3-4-5 triangle → perimeter 12 m.*

3.10 Circumference of a Circle

(Math 10-27) $$C = 2\pi r$$

Where,
C = Circumference (m)
r = Radius (m)

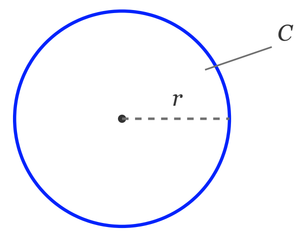

Figure 3.10 — circumference of a circle.

$C = 2\pi r$ looks like a definition of circumference, but read it the other way and it is the definition of π. Take any circle at all — a coin, a grain silo, the equator — measure the distance around and divide by the distance across, and you get the same number every time. That constancy is not obvious and it is not free; it is a genuine property of flat space, and it fails on a sphere, where a circle drawn around the north pole has a circumference of less than π times its diameter measured along the surface. On paper it holds, so one number serves every circle, and $C = \pi d$ is the same statement written for the measurement you actually take.

A worked instance: a 3-inch nominal steel pipe has an outside diameter of 88.9 mm, so a strap around it reads $\pi \times 88.9 = 279$ mm. Run it backwards and you have the field trick — a flexible tape around any pipe, divided by π, gives the outside diameter without ever getting calipers onto it. A 700c bicycle wheel is about 2.10 m around, which is how a cycle computer turns wheel revolutions into distance, and why entering the wrong tyre size quietly skews every ride you record.

About π itself. Archimedes bracketed it around 250 BC by squeezing a circle between an inscribed and a circumscribed 96-sided polygon, getting $223/71 < \pi < 22/7$ — correct to two decimals and honest about its own uncertainty, which was remarkable work for the third century BC. It cannot be written down exactly. Lambert proved in 1761 that π is irrational, so no fraction equals it and no decimal expansion ever repeats, and Lindemann proved in 1882 that it is transcendental, meaning it satisfies no polynomial equation with whole-number coefficients. That second result is what finally killed squaring the circle, a construction problem people had chased for two thousand years.

Two places this goes wrong. The first, again, is **radius against diameter**: the formula wants r, you measured d, and forgetting to halve doubles your answer. Circumference is more forgiving than area here — the error is a factor of two rather than four — but it is still the wrong number of metres of insulation. The second is treating $22/7$ and 3.14 as interchangeable with π. The first is 0.04% high, the second 0.05% low, which is invisible on a fence line and unacceptable on a machined bore. Finally, notice that circumference scales *linearly* while area scales as the square: double the pipe and you use twice the lagging but four times the flow area. Wrapping a large tank is cheap per litre stored, which is the same reason big things lose heat more slowly than small ones.

Worked example 3.10. Given r = 1 m; find C (circumference). *r = 1 m → C = 2 pi = 6.28319 m.*

3.11 Ellipse Perimeter (Ramanujan Approximation)

(Math 10-28) $$P \approx \pi \left[3(a+b) - \sqrt{(3a+b)(a+3b)}\right]$$

Where,
P = Perimeter (m)
a = Semi-major axis (m)
b = Semi-minor axis (m)

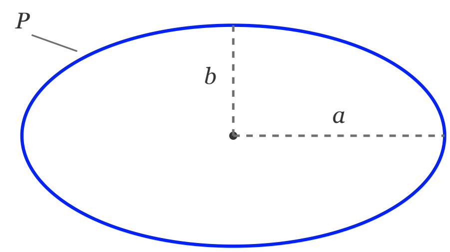

Figure 3.11 — ellipse perimeter (ramanujan approximation).

Here is one of mathematics' famous asymmetries: an ellipse's area is the effortless πab, but its perimeter has no elementary closed form at all. The exact answer is a complete elliptic integral of the second kind, which cannot be written with ordinary functions — so every practical formula is an approximation.

This one is Srinivasa Ramanujan's second approximation, published in 1914, and it is astonishingly good: for ellipses that are not extremely elongated its error is smaller than one part in a billion, far beyond any measurement you could take of a real object. Set $a = b$ and it returns $2\pi a$ exactly, the circle's circumference, as any honest approximation must.

The problem is not academic. Elliptical duct, oval tanks, running-track lanes and planetary orbits all need a perimeter, and Ramanujan's expression is what engineering handbooks quietly use.

Worked example 3.11. Given a = 1 m, b = 1 m; find P (perimeter). *a = b = 1 m reduces to a circle → 2 pi m.*

3.12 Circular Sector Area

(Math 10-29) $$A = \frac{1}{2} r^2 \theta$$

Where,
A = Sector area (m²)
r = Radius (m)
θ = Central angle (°)

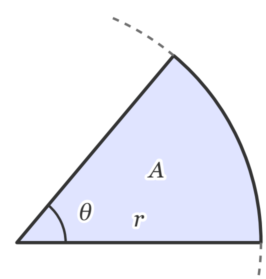

Figure 3.12 — circular sector area.

A sector is a pizza slice: the wedge cut from a circle by two radii. Its area is simply the slice's share of the whole pie — the fraction θ/(2π) of the full circle's πr² — which collapses to the tidy A = ½r²θ when the angle is in radians. Degree input is converted automatically, but the ½r²θ form itself only works in radians; plugging in degrees directly is the classic trap. The formula is meaningful for angles from 0 up to 2π, where the sector becomes the whole circle.

A concrete field application: an irrigation sprinkler set to sweep a 120° arc (θ = 2π/3 ≈ 2.094 rad) with a 10 m throw waters A = ½ × 10² × 2.094 ≈ 105 m² of lawn — just one-third of the 314 m² a full-circle sweep would cover, exactly as the slice picture predicts.

Worked example 3.12. Given r = 6 m, θ = 60 °; find A (sector area). *r = 6 m, 60 deg slice → 6 pi = 18.84956 m².*

3.13 Arc Length

(Math 10-30) $$s = r\theta$$

Where,
s = Arc length (m)
r = Radius (m)
θ = Central angle (°)

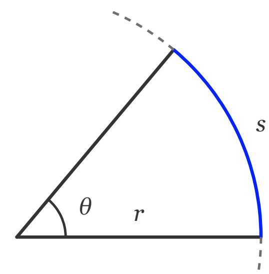

Figure 3.13 — arc length.

Arc length is radius times angle — and this little product is no coincidence, it is the very *definition* of the radian: one radian is the angle whose arc exactly equals the radius. That makes s = rθ the reason radians exist at all. The name was coined around 1870 by James Thomson (brother of Lord Kelvin), but the idea of measuring angles by arc goes back to Roger Cotes in 1714. The relation holds for angles from 0 up to 2π, where the arc becomes the full circumference 2πr. Enter degrees and formula.expert converts them to radians before multiplying.

Engineers lean on this constantly. A highway curve of radius 200 m that turns the road through 30° (0.524 rad) contains 200 × 0.524 ≈ 105 m of pavement; a gear tooth's working surface, a satellite's ground track, and the belt wrap on a pulley are all measured the same way.

Worked example 3.13. Given r = 10 m, θ = 90 °; find s (arc length). *r = 10 m, quarter turn (90 deg) → 5 pi = 15.70796 m.*

Module 3 practice

31. Rectangles and parallelograms — A community garden bed measures 8 m by 6 m. *Calculate the area of the bed.*

32. Rectangles and parallelograms — A parallelogram-shaped path sign has a base of 10 m. Its slanted side is 6 m long, and the perpendicular distance between its base and top is 4 m. *Calculate the area of the sign.*

33. Triangles and trapezoids — A triangular kite panel has an area of 28 m² on a base of 8 m. *Determine the height of the panel.*

34. Triangles and trapezoids — A trapezoid-shaped deck has parallel sides of 6 m and 8 m, set 6 m apart. *Calculate the area of the deck.*

35. Circles — A circular fountain has a radius of 15 m. A railing will run right around its edge. *Calculate the length of railing needed. (Take π = 3.14.)*

36. Circles — A circular fountain has a radius of 6 m. A railing will run right around its edge. *Calculate the length of railing needed. (Take π = 3.14.)*

37. Sectors and arcs — A pizza of radius 10 cm is cut into slices. One slice has a centre angle of 72°. *Determine the area of that slice. (Take π = 3.14.)*

38. Sectors and arcs — A Ferris wheel cabin sits 15 m from the hub. The wheel turns through 90° and stops. *Determine how far the cabin travelled. (Take π = 3.14.)*

39. Area in reverse — A rectangular banner has an area of 60 m² and a length of 10 m. *Determine the width of the banner.*

40. Area in reverse — A triangular pennant has an area of 40 m² on a base of 10 m. *Determine the height of the pennant.*

41. Composite floors — A living-room floor is a 9 m by 4 m rectangle, plus a triangular bay window nook with a base of 4 m and a depth of 4 m. *Determine the total floor area, piece by piece.*

42. Composite floors — A banquet-hall floor is a 16 m by 20 m rectangle with a semicircular stage end of radius 10 m on one end. (Take π = 3.14.) *Determine the total floor area, piece by piece.*

43. The Paint Estimate — Last job of the summer. A shed's end wall is 14 m wide and 5 m tall to the eaves, with a gable peak rising 4 m higher. A door 2 m tall and 2 m wide and a round window of radius 2 m stay unpainted. Paint costs $2 per square metre of wall. (π = 3 today.) Work each line — every answer feeds the next. *Determine what painting the wall will cost, one line at a time.*

44. The Paint Estimate — Bonus mark, read off the receipt: last month a single tin covered 10 m² of fence and cost $50. *Determine the price per square metre that tin delivered.*

MODULE 4

Volume & Surface Area

Prisms · Cylinders · The one-third family · Spheres · Wrapping the box · Wrapping the curves · Litres and cubes · Volume in reverse · The Grain Silo

4.1 Rectangular Prism Volume

(Math 10-31) $V = l \cdot w \cdot h$

Where,

V = Volume (L)

l = Length (m)

w = Width (m)

h = Height (m)

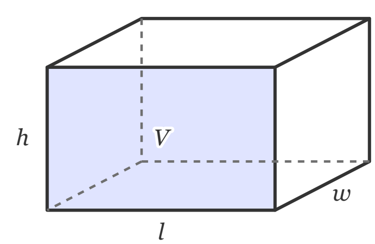

Figure 4.1 — rectangular prism volume.

Length times width times height is less a formula than the definition of volume made arithmetic. Lay unit cubes across the floor of the box, l of them one way, w the other, so lw cubes in a single layer — then stack h layers. Counting them *is* the volume, and $V = lwh$ is that count written down. Every other volume formula on this site is ultimately a way of getting back to this one, by slicing an awkward solid into pieces that behave like boxes.

A worked instance from a job site: a strip footing 2.0 m long, 500 mm wide and 300 mm deep takes $2.0 \times 0.5 \times 0.3 = 0.30$ m³ of concrete. Each dimension recovers by division, so a 0.30 m³ pour spread 100 mm deep over a 2.0 m length must be 1.5 m wide.

The conversion worth memorising lives here. One cubic metre is exactly **1000 litres**, because a litre is a cube 100 mm on a side and a thousand of those fill a metre cube. So a rectangular tank 1.5 m by 0.8 m filled to 1.1 m holds 1.32

m³, which is 1320 litres — and that litre figure is the one you need before dosing anything into it at so many millilitres per litre. In imperial the same step runs through 1 ft³ = 7.48 US gallons.

The mistakes here are all about units, and they are expensive. The first is **mixing units between the three dimensions**, which is almost guaranteed when a slab is quoted in metres but its thickness in millimetres. A pad 6 m by 4 m by 100 mm is 2.4 m³; entered as 6 × 4 × 100 it reads 2400, a thousandfold error that looks like a plausible number of something. Convert every dimension to one unit before multiplying, never afterwards. The second is the scale factor between cubic units: cm³ to m³ is a million, not a hundred, because the conversion is cubed along with the length. The third is applying this to something that is not a box. Real excavations batter outward, real rooms are out of square, and real tanks have dished ends; for those, split the shape into parts, or use an average dimension and accept that you now have an estimate rather than an answer.

Worked example 4.1. Given l = 2 m, w = 3 m, h = 4 m; find V (volume). *Box 2 x 3 x 4 m → V = 24 m3.*

4.2 Cube Volume

(Math 10-32) $V = a^3$

Where,

V = Volume (L)

a = Edge length (m)

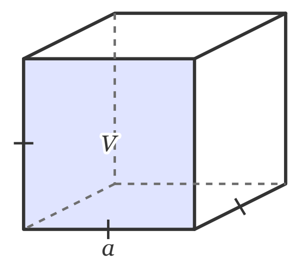

Figure 4.2 — cube volume.

A cube's volume is its edge multiplied by itself three times, $V = a^3$, and the arithmetic operation takes its name from the shape rather than the other way round — we say "cubing" because this is what it does. A shipping crate 1.2 m on each edge holds $1.2^3 = 1.73$ m³. The most useful instance is smaller: a cube 100 mm on a side holds exactly one litre, which is where the litre comes from and the quickest way to picture any volume you are handed.

Going backwards, $a = \sqrt[3]{V}$. Unlike the square root, the real cube root is single-valued and happily accepts negatives, so there is no sign ambiguity to resolve — a 20 L cubic container has edges of $\sqrt[3]{0.020} = 0.271$ m, and that is the only answer.

That inverse has a famous history. The Delian problem asked for the edge of a cube with twice the volume of a given one — legend has it the oracle at Delos demanded an altar of doubled size to end a plague, and the Athenians doubled every edge instead, producing an altar eight times too big. The required number is $\sqrt[3]{2} \approx 1.2599$, and the Greek question was whether it could be constructed with compass and straightedge alone. It cannot. Pierre Wantzel proved it impossible in 1837, more than two thousand years after the question was asked.

The Athenians' mistake is still the one people make, and it is worth stating plainly: **volume scales as the cube of size**. Double every dimension and you get eight times the contents, not two — a tank twice as wide, twice as long and twice as deep holds eight times the water and, if it is full, weighs eight times as much, which is a structural problem and not a rounding error. Run it the other way and the surprise reverses: to double a container's capacity you need to grow each edge by only 26%. "Twice as big" is a phrase with no fixed meaning, and asking which sense is intended is usually worth doing. The other trap is unit conversion, where the same exponent bites: 1 m³ is 1,000,000 cm³ and 1000 litres, and cubic units convert by the cube of the linear factor every time.

Worked example 4.2. Given a = 2 m; find V (volume). *Cube a = 2 m → V = 8 m3.*

4.3 Triangular Prism Volume

(Math 10-33) $V = \frac{1}{2}bh_tL$

Where,

V = Volume (L)

b = Triangle base (m)

h_t = Triangle height (m)

L = Prism length (m)

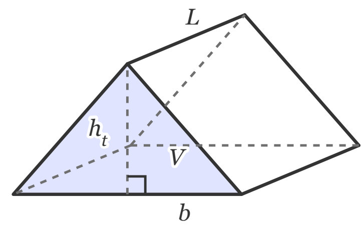

Figure 4.3 — triangular prism volume.

A triangular prism is a triangle extruded along a straight line: a tent, a gable roof space, a wedge of cheese, a section of dredged spoil. Its volume is the triangular end area $\frac{1}{2}bh_t$ multiplied by the run L. The subscript matters, because h_t is the height of the *triangle*, measured perpendicular to its base, and has nothing to do with how tall the prism stands.

The formula is worth knowing for the attic. A house 10 m long with a gable 8 m wide rising 3 m to the ridge encloses $\frac{1}{2}(8)(3)(10) = 120$ m³ of air, which is exactly the number an insulation contractor or a ventilation designer needs. A handy check on your work: a 3-4-5 right triangle has area exactly 6, so any prism built on one should come out as six times its length.

Worked example 4.3. Given b = 3 m, h_t = 4 m, L = 2 m; find V (volume). *3-4-5 triangle end, 2 m long → 12 m3.*

4.4 Prism Volume (General Cross-Section)

(Math 10-34) $V = BL$

Where,

V = Volume (L)

B = Cross-sectional area (m²)

L = Length (m)

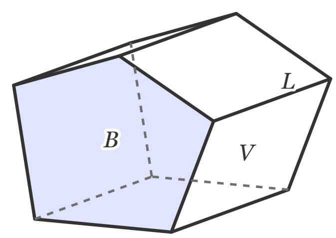

Figure 4.4 — prism volume (general cross-section).

This is the parent formula that every other prism volume is a special case of. If the cross-section does not change along the length, the volume is simply cross-section times length, $V = BL$. Feed it $B = \pi r^2$ and you get the cylinder. Feed it $B = lw$ and you get the box. Feed it a triangle, an I-beam profile, a corrugated sheet, or the messy irregular outline of a river channel, and it still works.

The reason it works even for shapes that lean is Cavalieri's principle, named for Bonaventura Cavalieri, who argued in 1635 that two solids of the same height with equal cross-sections at every level have the same volume. That is why an oblique prism holds exactly as much as an upright one of the same base and height, and why a leaning stack of coins occupies the volume of a neat one.

The unit bookkeeping is where people slip. B is an area and L is a length, so a cross-section in square inches multiplied by a length in feet gives nothing meaningful until you reconcile them. The solver handles that conversion for you, but the habit of checking dimensions by hand is worth keeping.

Worked example 4.4. Given V = 1 ft³, L = 6 in; find B (cross-sectional area). *1 ft3 over a 6 in run → cross-section 2 ft² (0.18580608 m2).*

4.5 Cylinder Volume

(Math 10-35) $$V = \pi r^2 h$$

Where,
V = Volume (L)
r = Radius (m)
h = Height (m)

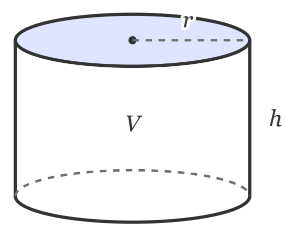

Figure 4.5 — cylinder volume.

A cylinder is a circle extruded through a height, so its volume is the base area πr² times h. It is the workhorse formula for anything round and straight-sided: storage tanks, pipes, drums, boreholes, and cans. Because the radius enters squared, diameter mistakes are costly — using a diameter where a radius belongs inflates the answer four-fold, the most common error in tank sizing.

A field example from water treatment: a vertical tank 1.2 m in diameter (r = 0.6 m) filled to 2.5 m holds V = π(0.6)²(2.5) ≈ 2.83 m³, about 2,830 litres — the number you need before dosing treatment chemicals at so many millilitres per litre. Solving for r takes the principal positive square root, the only physical radius; solving for h is a plain division. For a horizontal cylindrical tank that is only partly full, this formula gives total capacity, but the partial volume needs the circular-segment geometry instead.

Worked example 4.5. Given r = 1 m, h = 2 m; find V (volume). *Cylinder r = 1 m, h = 2 m → V = 2 pi m3.*

4.6 Cone Volume

(Math 10-36) $$V = \frac{1}{3}\pi r^2 h$$

Where,
V = Volume (L)
r = Base radius (m)
h = Height (m)

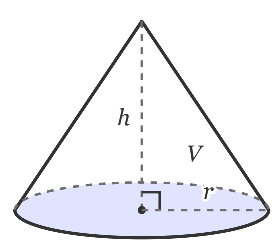

Figure 4.6 — cone volume.

A cone holds exactly one-third the volume of the cylinder that encloses it — the same one-third rule that governs pyramids and every solid that tapers linearly to a point. Democritus guessed the ratio around 400 BC, and Eudoxus proved it rigorously with the method of exhaustion, a forerunner of integral calculus. The h in the formula is the *perpendicular* height from base to apex, not the slant height along the side; confusing the two overstates the volume.

A worked example: a conical stockpile of sand 6 m across (r = 3 m) and 2 m tall contains V = (1/3)π(3)²(2) ≈ 18.85 m³ — roughly two truckloads, which is exactly how aggregate yards estimate inventory from a tape measure and a clinometer. Solving for r takes the principal positive square root, and solving for h is a plain division, so both inversions are single-valued for physical inputs.

Worked example 4.6. Given r = 3 m, h = 4 m; find V (volume). *Cone r = 3 m, h = 4 m → V = 12 pi m3.*

4.7 Pyramid Volume

(Math 10-37) $$V = \frac{1}{3}Bh$$

Where,

V = Volume (L)

B = Base area (m²)

h = Height (m)

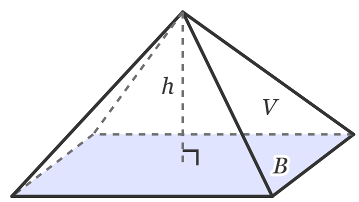

Figure 4.7 — pyramid volume.

Any pyramid fills exactly one-third of the prism that shares its base and height. Not roughly a third — exactly, and for any base shape at all. You can see the square case with your hands: three identical pyramids, each with the base of a cube and its apex at one of the cube's top corners, fit together to fill that cube with nothing left over. The general case follows from Cavalieri's principle. Slice both solids at any height and compare the cross-sections; if they match at every level, the volumes match. At a fraction x of the way down from the apex, a pyramid's cross-section is a scaled copy of the base with linear scale x, so its area is x^2B. Add those up from apex to base — which is to say integrate x^2 from 0 to 1 — and you get $\frac{1}{3}$. The one-third is the integral of a square, which is also why cones share it.

Because only the cross-sections matter, leaning the apex sideways changes nothing. An oblique pyramid holds exactly what the upright one of the same base and height holds, the same way a leaning stack of paper holds the same paper.

A worked instance: the tapered hopper under a feed bin, square at 1.2 m on a side and drawing down to a point 900 mm below, has $B = 1.44$ m² and $V = \frac{1}{3}(1.44)(0.9) = 0.43$ m³, about 430 litres. And the monumental one — the Great Pyramid at Giza, 230 m on a side and originally 146.6 m tall, encloses $\frac{1}{3}(52{,}900)(146.6) \approx 2.6$ million m³ of stone.

Two errors, and the first is specific to this page. B is the base **area**, not the base edge. Typing 230 for the Great Pyramid instead of 52,900 understates the answer by a factor of 230, and because the field accepts any number it will not object. Compute the base area first, in whatever shape the base actually is. The second error is the height: h is measured straight up from the base plane to the apex, never along the slope of a face. At Giza the slant height up the middle of a face is 186 m against a true height of 146.6 m, so using it would inflate the volume by 27%. The slant is the number you can put a tape on, which is exactly why it keeps getting entered. If the slant height l and the base half-width a are what you have, recover the true height with $h = \sqrt{l^2 - a^2}$ before coming back here.

Worked example 4.7. Given B = 6 m², h = 2 m; find V (volume). *Pyramid B = 6 m2, h = 2 m → V = 4 m3.*

4.8 Sphere Volume

(Math 10-38) $$V = \frac{4}{3}\pi r^3$$

Where,
V = Volume (L)
r = Radius (m)

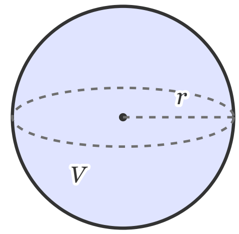

Figure 4.8 — sphere volume.

The volume of a sphere is four-thirds π times the radius cubed. Archimedes considered this his finest result: a sphere fills exactly two-thirds of the smallest cylinder that contains it, and he asked that the figure be carved on his tombstone. Because volume scales with the *cube* of the radius, small changes in size have outsized effects — doubling a balloon's radius gives it eight times the volume of gas.

A worked example: a spherical propane tank with a radius of 1.5 m holds V = (4/3)π(1.5)³ ≈ 14.14 m³, or about 14,100 litres. Going the other way, solving for r takes a cube root, which is single-valued for real numbers — no sign ambiguity to worry about. The same cubic scaling explains why raindrops, planets, and bubbles are dominated by their largest members: most of the total volume lives in the biggest few.

Worked example 4.8. Given r = 3 m; find V (volume). *Sphere r = 3 m → V = 36 pi m3 = 113097.34 L.*

4.9 Sphere Surface Area

(Math 10-39) $$S = 4\pi r^2$$

Where,
S = Surface area (m²)
r = Radius (m)

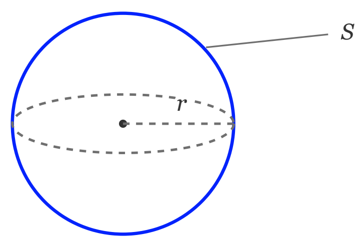

Figure 4.9 — sphere surface area.

A sphere's surface area is exactly four times the area of its great circle — the disc you get by slicing it through the centre. Archimedes proved this with an elegant argument comparing the sphere to its circumscribing cylinder; in modern language, S = 4πr² is simply the derivative of the volume (4/3)πr³ with respect to r, because growing a sphere adds a thin shell whose volume is surface area times thickness.

A worked example: to paint a hemispherical dome of radius 10 m, the curved surface is half a sphere, S = ½ · 4π(10)² ≈ 628 m² — at 8 m² per litre you would budget roughly 79 litres per coat. Solving for r takes the principal (positive) square root, the only physically meaningful choice since a radius is a positive length. The square-law scaling is why heat loss, drag, and paint budgets all grow four-fold when the radius merely doubles.

Worked example 4.9. Given r = 0.5 m; find S (surface area). *Sphere r = 0.5 m → S = pi m2.*

4.10 Rectangular Prism Surface Area

(Math 10-40) $$A = 2(lw + lh + wh)$$

Where,

A = Surface area (m²)
l = Length (m)
w = Width (m)
h = Height (m)

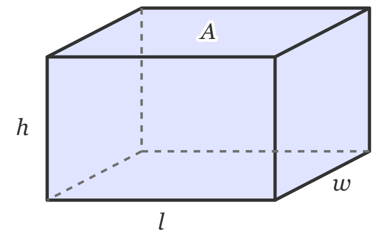

Figure 4.10 — rectangular prism surface area.

A box has six faces in three matching pairs, so the total area is $2(lw + lh + wh)$ — top and bottom, front and back, two ends. It is the formula behind every cardboard-cost estimate, every quote for cladding a duct, and every calculation of how much paint a shipping crate will drink. A carton 40 cm × 30 cm × 20 cm needs $2(1200 + 800 + 600) = 5200$ cm² of board, before you add a single glue flap.

The sanity check to keep in your pocket: set $l = w = h = s$ and the expression collapses to $6s^2$, the cube. If a general box formula does not reduce to six squares when you feed it a cube, it is wrong.

Solving backwards for one edge stays easy, which surprises people who expect a quadratic. Peel off the two faces that do not involve the unknown edge and what remains is linear: $A - 2wh = 2l(w + h)$. The catch is physical rather than algebraic. If the area you were given is less than those two fixed faces already account for, no box exists, and the solver says so rather than handing back a negative length.

Worked example 4.10. Given l = 3 m, w = 3 m, h = 3 m; find A (surface area). *3 m cube as a prism → 54 m² (= 6 s²).*

4.11 Cube Surface Area

(Math 10-41) $$S = 6a^2$$

Where,

S = Surface area (m²)
a = Edge length (m)

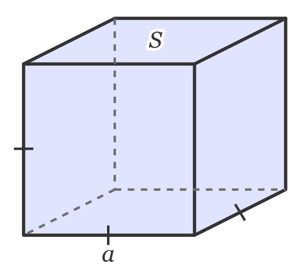

Figure 4.11 — cube surface area.

Six identical square faces, so $S = 6a^2$. Painting a cubic tank 2 m on a side means covering $6 \times 2^2 = 24$ m², which at 10 m² per litre is about 2.4 litres per coat. Backwards, $a = \sqrt{S/6}$ takes the principal positive root, an edge length being positive by definition: 54 m² of sheet metal folds into a cube 3 m on a side.

Set this beside $V = a^3$ and something important falls out. The ratio of surface to volume is $S/V = 6/a$, which **shrinks as the cube grows**. Skin goes up with the square of size while contents go up with the cube, so a big object has proportionally less outside for its inside. That single fact explains a long list of otherwise unrelated observations: why a large block of ice outlasts the same mass in cubes, why bulk storage costs less per litre to insulate, why a mouse must eat constantly while an elephant does not, and why cells stay microscopic — past a certain size, a cell cannot get nutrients across its membrane fast enough to supply its own interior.

The cube also holds a modest optimisation prize. Of all rectangular boxes with a given volume, the cube has the least surface area, so it is the cheapest box to build, to clad and to heat. A sphere beats it outright — that is why bulk gas is stored in spheres — but among shapes you can make from flat sheet and square corners, the cube wins.

Three things go wrong. The first is counting faces you do not have. A tank sitting on a slab presents five faces to the paintbrush, not six; an open-topped bin has five as well but a different five; a form for casting has an inside and an outside. Ask which surfaces actually exist before multiplying by 6. The second is the unit exponent: areas convert by the square of the length factor, so 1 m² is 10,000 cm², not 100, and an edge given in centimetres with an answer wanted in square metres needs converting first. The third is expecting the surface area to scale with the volume — it does not, and the whole previous paragraph is about why. Doubling the edge quadruples the paint and octuples the contents, so a coating estimate borrowed from a smaller tank of the same shape will always run high per litre stored.

Worked example 4.11. Given a = 1 m; find S (surface area). *Cube a = 1 m → S = 6 m2.*

4.12 Cylinder Surface Area

(Math 10-42) $$S = 2\pi r^2 + 2\pi r h$$

Where,

S = Total surface area (m²)

r = Radius (m)

h = Height (m)

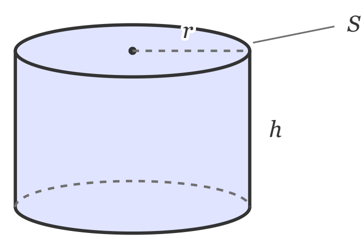

Figure 4.12 — cylinder surface area.

Take a can, cut the two ends off, slit the side and lay it flat. What you have is two discs and one rectangle — and the rectangle's width is the distance the cut travelled around the can, which is the circumference $2\pi r$. That is the whole derivation: $S = 2\pi r^2 + 2\pi r h$, two caps plus a label. The second term is called the lateral or curved surface area, and it is the one that does most of the work in practice, because it is the part that grows with the height.

A worked instance in units you would meet. A vertical hot water storage tank 600 mm in diameter and 1.5 m tall has $r = 0.3$ m, so the caps come to $2\pi(0.3)^2 = 0.57$ m² and the side to $2\pi(0.3)(1.5) = 2.83$ m², totalling 3.39 m². That is the jacket area, and multiplied by the temperature difference and divided by the insulation's R-value it is the tank's standing heat loss. Backwards, $h = S/(2\pi r) - r$ tells you how tall a tank of a given radius must be to reach a required surface — the subtraction of r is the caps being paid for first.

Read alongside the volume $V = \pi r^2 h$, this formula answers a design question. For a fixed volume, the surface area is smallest when $h = 2r$ — when the can is exactly as tall as it is wide. That is the shape that uses the least metal per litre, and it is not the shape of the can in your cupboard, because the ends need thicker stock and a seaming operation the wall does not, so real cans are made taller and narrower than the geometry alone would suggest. Cost per square centimetre is not uniform, and the optimum shifts accordingly.

Three ways the answer comes out wrong. The first is the perennial one: **diameter entered as radius**, which quadruples the caps and doubles the side. The second is not asking which surfaces actually exist. This formula is for a *closed* cylinder. An open-topped drum has one cap, not two, so subtract πr^2; a bare length of pipe has no caps at all and is $2\pi rh$ alone; and a pipe you intend to coat inside and out has two curved surfaces at slightly different radii. The third shows up when solving for height: if the surface area you enter is less than the two end caps could account for on their own, no positive height exists and the solver says so rather than returning a negative tank.

Worked example 4.12. Given r = 1 m, h = 1 m; find S (total surface area). *Closed can r = 1 m, h = 1 m → S = 4 pi m2.*

4.13 Cylinder Lateral Surface Area

(Math 10-43) $$A = 2\pi rh$$

Where,

A = Lateral surface area (m²)
r = Radius (m)
h = Height (m)

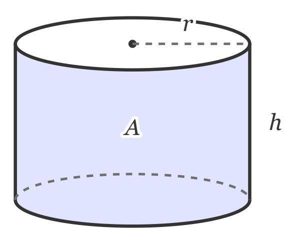

Figure 4.13 — cylinder lateral surface area.

Cut a cylinder's wall along a vertical line and roll it flat and you get a rectangle: $2\pi r$ wide, h tall, so $A = 2\pi rh$. No ends, no caps, just the wrap. This is the paper label on a soup can, the sheet metal around a round duct, the pipe insulation you order by the square metre, and the area a painter charges for on a storage silo.

Keep it distinct from the closed cylinder's total surface $2\pi r^2 + 2\pi rh$, which adds the two circular ends. For a squat shape the difference is enormous. A cylinder with $r = 1$ m and $h = 0.2$ m has a wall of only 1.26 m² but end caps totalling 6.28 m², so quoting the total when you meant the wrap overstates the job by a factor of six. For a long thin pipe the ends are negligible and the two numbers converge.

The inverse is the one field crews actually reach for: given a roll of jacket of known area, how far along the pipe will it get you? That is $h = A/(2\pi r)$, and it is why insulation is sold by area rather than by length.

Worked example 4.13. Given r = 1 m, h = 1 m; find A (lateral surface area). *r = 1 m, h = 1 m → 2 pi m² (wall only).*

4.14 Cone Lateral Surface Area

(Math 10-44) $$A = \pi r l$$

Where,

A = Lateral surface area (m²)
r = Base radius (m)
l = Slant height (m)

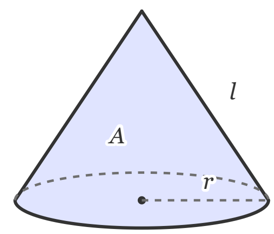

Figure 4.14 — cone lateral surface area.

Cut a paper cone up one side and roll it flat and you get a sector of a circle: a pie slice whose radius is the slant height l and whose curved edge is the base circumference $2\pi r$. A sector's area is half its arc times its radius, so $A = \frac{1}{2}(2\pi r)(l) = \pi r l$. That is the whole derivation, and it is why sheet-metal workers can lay out a cone as a flat pattern before they ever bend anything.

The formula runs the trade backwards just as often. A fabricator with a fixed sheet size divides the available area by πr to find the longest cone that can be cut from it, and a supplier quoting a run of conical hoppers prices the job by square metres of plate. Note what is missing: no base. This is the skin of the side only, which is exactly what you want for a witch's hat roof, a loudspeaker cone or a filter element.

The classic mistake is feeding in the vertical height instead of the slant height. They are different numbers, and the slant is always the larger of the two. A cone 3 m in radius and 4 m tall has a slant height of 5 m, so using 4 would undercount the metal by a fifth.

Worked example 4.14. Given r = 3 m, l = 5 m; find A (lateral surface area). *Cone r = 3 m, l = 5 m → 15 pi m2.*

4.15 Cone Slant Height

(Math 10-45) $$l = \sqrt{r^2 + h^2}$$

Where,

l = Slant height (m)
r = Base radius (m)
h = Vertical height (m)

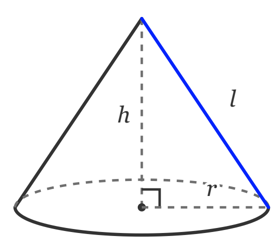

Figure 4.15 — cone slant height.

Slice a right circular cone straight down through its apex and the cut face is an isosceles triangle. Half of it is a right triangle with legs r and h and hypotenuse l, so Pythagoras hands over $l = \sqrt{r^2 + h^2}$ with no further work. Every cone formula that mentions surface uses l; every cone formula that mentions volume uses h, and this is the bridge between them.

In practice you measure whichever is easier and convert. A tape laid up the outside of a silo roof gives you the slant directly. A laser measuring the peak above the eaves gives you the vertical height. Roofers, tent makers and conveyor designers all live on this conversion, and the useful sanity check is that l must always exceed both r and h. If a supplied slant height comes out shorter than the radius, someone has mixed up which measurement is which.

Worked example 4.15. Given r = 3 m, h = 4 m; find l (slant height). *r = 3 m, h = 4 m → slant 5 m.*

4.16 Hemisphere Volume

(Math 10-46) $$V = \frac{2}{3}\pi r^3$$

Where,
V = Volume (L)
r = Radius (m)

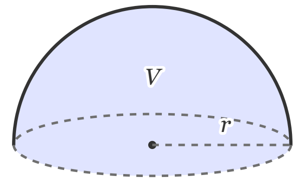

Figure 4.16 — hemisphere volume.

Half a sphere holds half a sphere's worth, so $V = \frac{2}{3}\pi r^3$. What makes it more than a trivial halving is the fact Archimedes proved and asked to have engraved on his tombstone: a sphere occupies exactly two thirds of the cylinder that just contains it. Cut everything in half and the hemisphere occupies two thirds of its enclosing cylinder too, which is where the $\frac{2}{3}$ in front of πr^3 literally comes from.

Practically it is the bowl formula and the dome formula. A mixing bowl 30 cm across holds $\frac{2}{3}\pi(0.15)^3 \approx 7.07$ litres brim full, which is why a "10 litre" bowl is larger than most people expect. It is also the capacity of a hemispherical tank head, the displacement of a dome-topped piston crown, and the earth volume in a hemispherical excavation.

Worked example 4.16. Given r = 3 m; find V (volume). *Hemisphere r = 3 m → 18 pi m3, half the 36 pi sphere.*

4.17 Hemisphere Total Surface Area

(Math 10-47) $$A = 3\pi r^2$$

Where,
A = Total surface area (m²)
r = Radius (m)

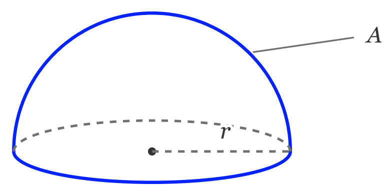

Figure 4.17 — hemisphere total surface area.

A solid hemisphere has two surfaces, and this formula counts both. The curved half-sphere contributes $2\pi r^2$, exactly half the sphere's $4\pi r^2$, and the flat disc that closes it off contributes πr^2. Together that is $A = 3\pi r^2$. Miss the base and you are a full third short.

So the first question to ask is always which surface you actually mean. Roofing a geodesic dome or gilding the outside of a bowl uses the curved $2\pi r^2$ alone, because the flat face is open air or is bolted to something. Chrome-plating a solid half-ball, costing the wrapper on a dome-shaped chocolate, or computing the total heat-transfer area of a hemispherical tank head that sits on a flat plate all need the full $3\pi r^2$. Many textbooks call $2\pi r^2$ the "curved surface area" and $3\pi r^2$ the "total surface area", and marks are lost on that distinction every exam season.

The curved part hides a lovely fact of its own. Archimedes showed that the sphere's surface area equals that of the cylinder wrapped around it, so a hemisphere's dome has exactly the same area as the label on a can of radius r and height r. Wrapping paper for a dome and wrapping paper for that can come to the same square metres.

Worked example 4.17. Given r = 1 m; find A (total surface area). *Solid hemisphere r = 1 m → 3 pi m2 (2 pi curved + pi base).*

Module 4 practice

45. Prisms — A storage locker measures 3 m long, 2 m wide and 3 m tall. *Calculate the volume of the locker.*

46. Prisms — A cube-shaped shipping crate has edges 5 m long. *Calculate the volume of the crate.*

47. Cylinders — A cylindrical rain tank holds 628.3 m³ and stands 8 m tall. *Determine the radius of the tank.*

48. Cylinders — A cylindrical water tank has a radius of 3 m and stands 6 m tall. *Calculate the volume of water the tank can hold.*

49. The one-third family — A conical pile of gravel at a landscaping yard has a base radius of 3 m and stands 6 m tall. *Calculate the volume of gravel in the pile.*

50. The one-third family — A glass pyramid skylight has a base of area 18 m² and rises 2 m to its point. *Calculate the volume of space under the skylight.*

51. Spheres — A weather balloon, fully inflated has a radius of 5 m. *Calculate the volume of helium inside the balloon.*

52. Spheres — A weather balloon, fully inflated has a radius of 5 m. *Calculate the surface area of the balloon's skin.*

53. Wrapping the box — A plywood storage chest measures 3 m by 2 m by 1 m. Every face gets a coat of paint — top, bottom and all four sides. *Calculate the total surface area to paint.*

54. Wrapping the box — A cube-shaped garden planter has edges 3 m long. All six faces are to be sealed with waterproof stain. *Calculate the surface area to seal.*

55. Wrapping the curves — A soup can has a radius of 4 cm and stands 10 cm tall. Its paper label covers the curved wall exactly — no overlap, and nothing on the lids. *Calculate the area of the label.*

56. Wrapping the curves — A closed steel drum has a radius of 9 cm and a height of 18 cm. The whole outside gets painted — wall, lid and base. *Calculate the total surface area to paint.*

57. Litres and cubes — A rectangular lunch cooler measures 35 cm by 20 cm by 10 cm on the inside. *Determine how many litres the cooler holds.*

58. Litres and cubes — A cylindrical rain cistern has a radius of 2 m and a water depth of 2 m. *Determine how many litres of water the cistern holds.*

59. Volume in reverse — A cube-shaped cold-storage room has a volume of 343 m³. *Determine the length of one edge of the room.*

60. Volume in reverse — A cylindrical grain bin holds 251.3 m³. Its radius is 4 m. *Determine the height of the bin.*

61. The Grain Silo — Harvest week's last job. A grain silo is a cylinder of radius 3 m with a wall 10 m tall, topped by a dome — a perfect half-sphere of the same radius. No calculator: keep π as a symbol and give each answer as a clean number times π. Every answer feeds the next. *Determine the silo's total capacity, one line at a time — leave π in every answer.*

62. The Grain Silo — The paint crew's turn. Their silo has a radius of 6 m and a cylindrical wall 10 m tall under its half-sphere dome. Only what shows gets painted: the wall and the dome's curve — no floor, no seams. π stays a symbol until the very last line. *Determine the painted area, then put a number on it with π ≈ 3.*

MODULE 5

Right-triangle trigonometry

Labelling the sides · Pythagoras · Choosing the ratio · Finding a side · Finding the angle · Elevation and depression · Grades and slopes · The Ramp Inspection

5.1 Right-Triangle Sine Ratio (SOH)

(Math 10-48) $$\sin\theta = \frac{o}{h}$$

Where,
θ = Acute angle (°)
o = Opposite side (m)
h = Hypotenuse (m)

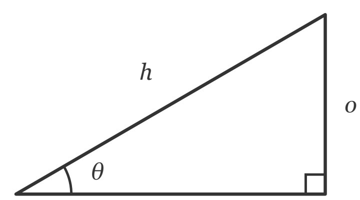

Figure 5.1 — right-triangle sine ratio (soh).

SOH — **S**ine is **O**pposite over **H**ypotenuse — is the first line of the most durable mnemonic in mathematics. In a right triangle, the sine of an acute angle is fixed by the triangle's shape alone: every right triangle with a 30° angle has an opposite side exactly half its hypotenuse, no matter the size. That constancy is what makes the ratio a tool. A field example: a 20 m guy wire anchored at 30° to the ground reaches a height of 20 × sin 30° = 10 m up the mast. The idea is ancient — Indian astronomers tabulated the *jya* (half-chord) around 500 CE, and a translation detour through Arabic and Latin gave us the word *sine*.

Solving for the angle uses the inverse function: θ = arcsin(o/h). The solver returns the principal branch only, which here is exactly right — the non-right angles of a right triangle are always acute, so the answer between 0° and 90° is the only valid one. The ratio o/h must stay below 1, since a leg can never outgrow the hypotenuse.

Worked example 5.1. Given θ = 30 °, h = 10 m; find o (opposite side). *30° angle, 10 m hypotenuse → 5 m opposite side.*

5.2 Right-Triangle Cosine Ratio (CAH)

(Math 10-49) $$\cos\theta = \frac{a}{h}$$

Where,
θ = Acute angle (°)
a = Adjacent side (m)
h = Hypotenuse (m)

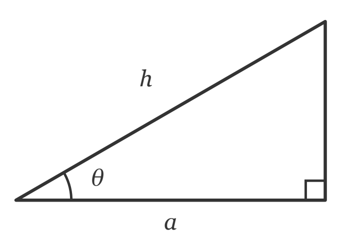

Figure 5.2 — right-triangle cosine ratio (cah).

CAH — **C**osine is **A**djacent over **H**ypotenuse. Like all the trigonometric ratios, it works because the shape of a right triangle is fixed by one acute angle alone: every right triangle with a 60° angle has an adjacent leg exactly half its hypotenuse, whether it is drawn on a napkin or laid out across a field. Similar triangles keep the ratio constant, and that constancy is what turns an angle into a length. The name is a contraction of *complementi sinus*, the sine of the complement, because $\cos\theta = \sin(90^\circ - \theta)$ — cosine is not a second idea but the same idea viewed from the other acute corner.

A worked instance you can check against a wall. A ladder is meant to stand at about 75° to the ground, the familiar one-out-for-four-up rule. A 6 m ladder therefore has its feet $6\cos 75^\circ = 1.55$ m from the base of the wall, and reaches 5.8 m up it. Run the other way, $\theta = \arccos(a/h)$: a 6 m ladder set 2.0 m out is standing at $\arccos(0.333) = 70.5^\circ$, flatter than it should be.

Three values are worth knowing cold: $\cos 0^\circ = 1$, $\cos 60^\circ = 0.5$, $\cos 90^\circ = 0$. Beyond the triangle, cosine is the universal "how much of this points that way" operator. The component of a force along a direction is $F\cos\theta$; the useful part of an alternating current is the power factor $\cos\varphi$; the projection of any vector onto any axis is a cosine. On the unit circle it is simply the x-coordinate, and the identity $\sin^2\theta + \cos^2\theta = 1$ is the Pythagorean theorem on a triangle of hypotenuse 1.

Where it goes wrong. The most frequent error is picking the wrong leg: the adjacent side is the one touching the angle that is *not* the hypotenuse — and since the hypotenuse also touches the angle, that phrasing is exactly where people slip. The second is degree-versus-radian mode. This solver handles the conversion, but a phone calculator left in radians returns $\cos(35) = -0.903$, a negative number where a positive one belongs, which is at least loud enough to notice. The third is a domain limit that is really a geometry lesson: a/h can never exceed 1, because a leg cannot outrun the hypotenuse, so an arccos that refuses to evaluate is telling you the two lengths do not form a right triangle. Finally, solving for the hypotenuse divides by $\cos\theta$, which collapses toward zero as the angle nears 90° — near-vertical geometry makes this rearrangement extremely sensitive to a small error in the angle.

Worked example 5.2. Given θ = 60 °, h = 8 m; find a (adjacent side). *60° angle, 8 m hypotenuse → 4 m adjacent side.*

5.3 Right-Triangle Tangent Ratio (TOA)

(Math 10-50) $$\tan\theta = \frac{o}{a}$$

Where,
θ = Acute angle (°)
o = Opposite side (m)
a = Adjacent side (m)

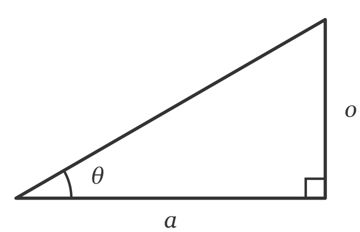

Figure 5.3 — right-triangle tangent ratio (toa).

TOA — **T**angent is **O**pposite over **A**djacent. It is the ratio to reach for when the hypotenuse is unknown, and in the field the hypotenuse usually *is* unknown, because it runs along your line of sight to something you cannot put a tape on. Tangent is not independent of the other two: dividing $\sin\theta = o/h$ by $\cos\theta = a/h$ cancels the hypotenuse and leaves o/a, which is the formula on this page and the reason the hypotenuse drops out of the problem.

Here is the fact that makes this the most quietly useful ratio of the three: **tangent is slope**. Rise over run is opposite over adjacent, so $m = \tan\theta$, and every grade, pitch and gradient you have ever seen is a tangent in disguise. A 6% road grade is $\arctan(0.06) = 3.43^\circ$. A 6:12 roof pitch is $\arctan(0.5) = 26.6^\circ$. A 1:12 ramp is 4.76°. If you have ever wondered why a highway sign warning of an 8% grade describes something that looks nowhere near eight degrees, this is why — and it is worth noticing that a 100% grade is 45°, not vertical.

The classic use is an angle of elevation. Stand 50 m out from a tower on level ground and sight the top at 31°: the rise above your eye is $50 \tan 31^\circ = 30.0$ m. Backwards, $\theta = \arctan(o/a)$, and the principal branch is exactly right here — two positive legs always land in 0°–90°, so there is no ambiguity to resolve.

Four cautions, in the order they cost people money. First, that tower is not 30 m tall. The formula returns the rise above the *instrument*, so you must add your eye height, or the tripod height, to get the real total — a systematic error of a metre and a half that stays invisible because the answer looks reasonable. Second, a is the *horizontal* distance. On sloping ground a taped distance runs along the slope and is longer than the run, which inflates the result; that is why survey instruments reduce slope distance to horizontal before anything else happens. Third, tangent has no ceiling — it runs to infinity at 90° — and grows viciously sensitive as it approaches. At 30° a half-degree error in the sighting shifts the height by about 2%; at 80° the same half degree shifts it by 5%, and at 85° by 10%. Standing farther back and sighting at a shallower angle is nearly always the more accurate measurement. Fourth, as ever, check whether your calculator is in degrees.

Worked example 5.3. Given θ = 30 °, a = 90 m; find o (opposite side). *30° elevation, 90 m from base → 51.9615 m height (30√3).*

5.4 Percent Grade from Rise and Run

(Math 10-51) $$G = \frac{100\,\Delta h}{L}$$

Where,

G = Grade (%)

Δh = Vertical rise (m)

L = Horizontal run (m)

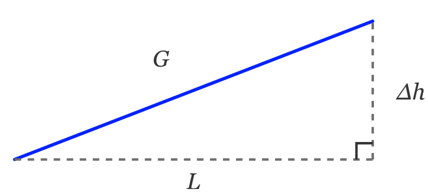

Figure 5.4 — percent grade from rise and run.

Grade is the trade's way of saying slope: rise over run, multiplied by a hundred so it reads as a percentage. The Romans understood the idea long before the notation existed — the aqueducts feeding the capital fell at grades near 0.02 %, and the Pont du Gard section drops about 2.5 cm per kilometre, a precision that still impresses. The trap is the denominator. Grade uses the *horizontal* run, not the sloped distance you would tape along the ground, and not the hypotenuse. On flat work the difference is invisible; on a 30 % haul road the sloped length is 4.4 % longer than the run, and quantities computed off the wrong one come back to bite at pay time.

A worked example: a parking lot drains 8 in over a 100 ft run. Convert first — 8 in is 0.667 ft — so G = 100 × 0.667 / 100 = 0.67 %, comfortably above the 0.5 % minimum most agencies allow for asphalt and below the 5 % that makes accessible routes non-compliant. Read backwards, the same formula sizes a ramp: to hold 5 % over a 3 ft rise you need 100 × 3 / 5 = 60 ft of run.

Worked example 5.4. Given G = 2 %, L = 250 m; find Δh (vertical rise). *2 % over 250 m → 5 m of rise.*

5.5 Grade to Slope Angle

(Math 10-52) $$\theta = \arctan\left(\frac{G}{100}\right)$$

Where,
θ = Slope angle (°)
G = Grade (%)

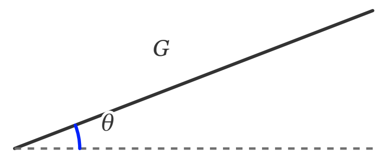

Figure 5.5 — grade to slope angle.

Percent grade and slope angle describe the same hill in two languages, and they are only interchangeable through the tangent — never by simple proportion. That is the single most common mistake on the topic: a 100 % grade is not vertical, it is 45°, because it rises one unit for every one it runs. Switzerland's Pilatus railway, the steepest cog line in the world, climbs at 48 %, which is a mere 25.6° — steep enough that the carriages are built as stepped terraces, but nowhere near the cliff that "48 percent" suggests to the untrained ear.

The two scales agree closely at small angles, which is why the confusion survives: at 5 % the angle is 2.86°, an error of under 0.1° if you naively read percent as degrees divided by nothing. By 20 % the grade angle is 11.31° and the shortcut is useless. A worked example: a haul road cut at 8 % sits at θ = arctan(0.08) = 4.57°; run it the other way and a 30° talus slope corresponds to G = 100 tan 30° = 57.7 %.

Worked example 5.5. Given G = 0.5 fraction; find θ (slope angle). *50 % grade (entered as 0.5) → 26.5651°.*

5.6 Slope Ratio (H:V) to Percent Grade

(Math 10-53) $$G = \frac{100}{n}$$

Where,
n = Horizontal units per 1 vertical
G = Equivalent grade (%)

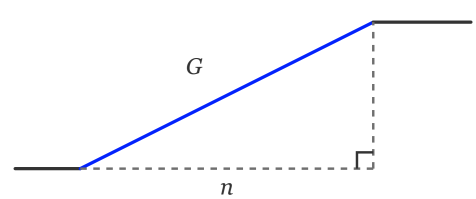

Figure 5.6 — slope ratio (h:v) to percent grade.

Earthwork drawings almost never label a cut face in percent. They call it 3:1 or 2:1, and the convention in North American civil practice is horizontal first — three metres out for every one up. Read it backwards and a 4:1 lawn slope becomes a 400 % cliff, which is how the argument usually starts on site. The conversion is a plain reciprocal: G = 100/n, so 2:1 is 50 %, 3:1 is 33.3 %, and 4:1 is 25 %. These are not arbitrary numbers — 3:1 is the flattest slope a ride-on mower handles safely, 2:1 is about the steepest that will hold topsoil and vegetation without armouring, and anything steeper generally wants riprap, geogrid or a wall.

A worked example: a detention pond is specified with 4:1 side slopes and the pond is 1.8 m deep, so each bank eats 4 × 1.8 = 7.2 m of horizontal room and the equivalent grade is

100/4 = 25 %. Note that geotechnical reports sometimes quote the same slope V:H instead — always check which number carries the "1".

Worked example 5.6. Given n = 3; find G (equivalent grade). *3:1 slope → 33.33 % grade.*

Module 5 practice

63. Pythagoras — A rectangular schoolyard measures 8 m by 15 m. A rope fence will run corner to corner, straight across. *Calculate the length of rope needed.*

64. Pythagoras — A rectangular schoolyard measures 5 m by 12 m. A rope fence will run corner to corner, straight across. *Calculate the length of rope needed.*

65. Finding a side — A grain-elevator conveyor climbs at a steep 60°, and its horizontal footprint measures 20 m along the ground. *Calculate the height the conveyor reaches.*

66. Finding a side — A zipline platform stands 8 m above the field, and the cable runs taut to the ground at 30°. *Calculate the length of the cable.*

67. Finding the angle — A loading ramp rises 3 m over a level run of 4 m. *Determine the angle the ramp makes with the ground.*

68. Finding the angle — A 5 m guy wire is anchored 4 m out from the base of its pole. *Determine the angle the wire makes with the level ground.*

69. Elevation and depression — From the top of a 60 m cliff, a lifeguard sights a swimmer at an angle of depression of 60°. *Determine how far the swimmer is from the base of the cliff.*

70. Elevation and depression — From a survey point 30 m from the base of a radio tower, the angle of elevation to its tip is 30°. *Calculate the height of the tower.*

71. Grades and slopes — A drainage embankment is cut at 5:1 — 5 m across for every 1 m down. *Calculate the equivalent percent grade.*

72. Grades and slopes — The site plan calls for a 50% embankment grade, but the crew sets their boards as an n:1 slope. *Determine the n of the equivalent n:1 slope.*

73. The Ramp Inspection — Final inspection of the season. A loading-dock ramp rises 9 m over a level run of 12 m, and the freight code caps a fixed ramp at 40°. Work each line — every answer feeds the next. *Determine the ramp's slope length, its grade, and its angle — then pass or fail it against the code.*

74. The Ramp Inspection — Second stop: the accessibility ramp at the office door rises 1 m over a level run of 20 m. The access code likes its ramps gentle, and its paperwork in both costumes. *Determine the ramp's grade, then write it as an n:1 slope.*

MODULE 6

Money & Interest

Reading the loan · Simple interest · Compound interest · Compounding frequency · The rule of 72 · Present value · Simple vs compound · The Savings Ledger

6.1 Simple Interest

(Math 10-54) $$I = P\,r\,t$$

Where,
I = Interest earned
P = Principal
r = Interest rate per period (decimal)
t = Number of periods

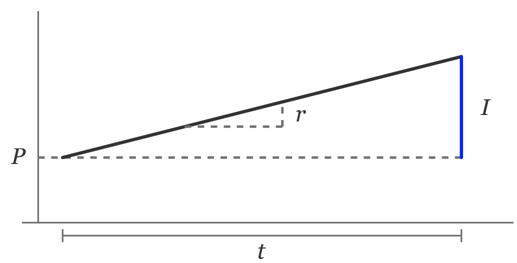

Figure 6.1 — simple interest.

Simple interest pays only on the original principal, never on past interest: a $2,000 loan at 6% per year for 3 years costs I = 2000 × 0.06 × 3 = $360, exactly $120 each year. Enter the rate as a decimal (6% → 0.06) and keep the rate and the time in the same period — a monthly rate with months, a yearly rate with years.

Short-term personal loans, car loans, and bonds' coupon payments often work this way. Compare with compound interest, where each period's interest joins the principal and the total pulls ahead of the simple-interest straight line.

Worked example 6.1. Given P = 1000, r = 5 %, t = 3 ; find I (interest earned). *$1000 at 5% for 3 periods → I = 150.*

6.2 Compound Interest (Periodic)

(Math 10-55) $$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Where,
A = Final amount
P = Principal
r = Interest rate per period (decimal)
n = Compounds per period
t = Number of periods

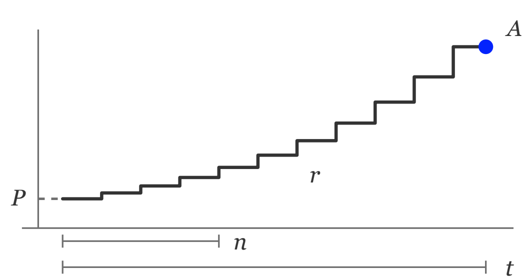

Figure 6.2 — compound interest (periodic).

With periodic compounding, each period's rate r is split into n slices and applied n times, so past interest starts earning interest of its own. $5,000 at 4% per year compounded monthly (n = 12) for 10 years grows to A = 5000 × (1 + 0.04/12)^{120} ≈ $7,454.16 — about $50 more than yearly compounding would give, because 120 small boosts beat 10 large ones.

Solving for r recovers the rate a savings product actually paid between two statements, and solving for t answers "how long until my balance reaches A?" Enter r as a decimal (4% → 0.04), and note that pushing n toward infinity lands on the continuous-compounding formula $A = Pe^{rt}$.

Worked example 6.2. Given P = 1000, r = 6 %, n = 12, t = 10 ; find A (final amount). *$1000 at 6%/yr monthly for 10 yr → A = 1819.40.*

6.3 Effective Annual Rate from a Nominal Rate

(Math 10-56) $$EAR = \left(1 + \frac{r}{m}\right)^m - 1$$

Where,
EAR = Effective annual rate
r = Nominal annual rate
m = Compounds per year

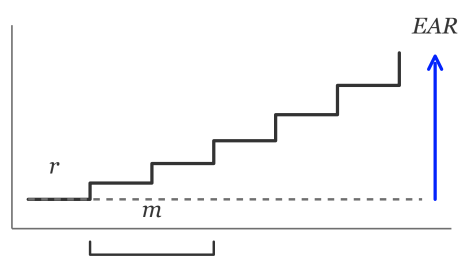

Figure 6.3 — effective annual rate from a nominal rate.

A quoted rate is not a cost until you know how often it compounds. "12% a year" charged monthly is really 1% twelve times, and $(1.01)^{12} - 1 = 12.68\%$. The extra 0.68 points is interest earned on interest, and it grows with the compounding frequency: the same 12% compounded daily comes to 12.747%, approaching the continuous limit $e^{0.12} - 1 = 12.75\%$.

This is the only fair way to compare two offers. A card at 19.99% compounded daily and a line of credit at 20.2% compounded annually are not what they appear, and the nominal figures rank them the wrong way round. Disclosure law exists precisely because of this gap, which is why lenders must publish an effective or annualised figure alongside the headline rate.

Going backwards recovers the nominal rate a lender must be quoting to produce a given effective one. Notice that m cannot be solved for: it sits in the base and the exponent at once, and no elementary rearrangement frees it.

Worked example 6.3. Given r = 12 %, m = 12 ; find EAR (effective annual rate). *12% nominal compounded monthly → 12.6825% effective.*

6.4 Rule of 72 (Doubling Time)

(Math 10-57) $$n \approx \frac{0.72}{i}$$

Where,
n = Periods to double
i = Rate per period

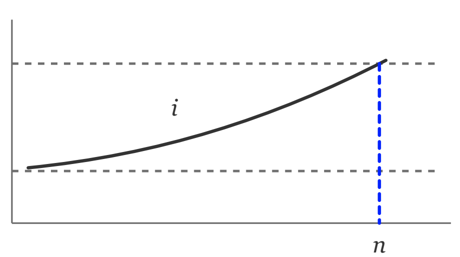

Figure 6.4 — rule of 72 (doubling time).

Divide 72 by the percentage rate and you have the years to double: 6% doubles in about twelve years, 9% in eight, 12% in six. It is the most useful piece of mental arithmetic in personal finance, and it has been in print since Luca Pacioli's *Summa de Arithmetica* of 1494, stated without proof as something merchants already knew.

The exact constant is not 72. Doubling requires $n\ln(1+i)=\ln 2$, and for small rates that is close to $0.693/i$, so 69.3 would be more accurate, and exactly right for continuous compounding. Seventy-two is used because it divides cleanly by 2, 3, 4, 6, 8, 9 and 12, and because the small upward fudge happens to compensate for the approximation across the range of rates people actually meet. It is at its best between about 6% and 10% and drifts noticeably above 20%.

The rule cuts both ways, which is the part worth remembering. At 3% inflation, prices double in 24 years, so a fixed pension halves in purchasing power over an ordinary retirement. The same arithmetic that makes savings look encouraging makes inflation look alarming.

Worked example 6.4. Given i = 6 %; find n (periods to double). *6% a year → doubles in about 12 years.*

6.5 Present Value

(Math 10-58) $$PV=\frac{FV}{(1+r)^t}$$

Where,
PV = Present value
FV = Future value
r = Discount rate per period (decimal)
t = Number of periods

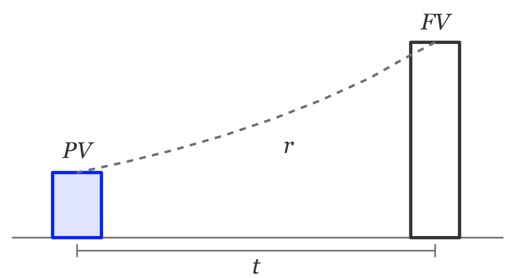

Figure 6.5 — present value.

Money later is worth less than money now, because money now could be invested. Present value runs compound growth in reverse: $10,000 arriving in 8 years, discounted at 5% per year, is worth PV = 10000 / 1.05^8 ≈ $6,768.39 today. Enter the rate as a decimal (5% → 0.05).

This one discount is the atom of finance — bond prices, mortgage balances, and a company's valuation are all sums of future cash flows each pulled back to today. Solving for r asks "what return does this deal imply?", and solving for t asks how long a target takes at a given rate.

Worked example 6.5. Given FV = 10000, r = 5 %, t = 8 ; find PV (present value). *$10,000 in 8 yr at 5% → PV = 6768.39.*

Module 6 practice

75. Simple interest — A student places $200 in a savings account paying 3% per year, simple interest, and leaves it untouched for 3 years. *Calculate the interest the account earns.*

76. Simple interest — A $1000 deposit at 2% per year, simple interest, has earned $60 of interest so far. *Determine how long the money has been invested.*

77. Compound interest — $2000 is invested at 10% per year, compounded annually, for 3 years. *Determine the amount in the account after 3 years.*

78. Compound interest — A savings account paying 5% per year, compounded annually, receives a deposit of $2000. The money is left alone for 2 years. *Calculate the balance at the end.*

79. Compounding frequency — $2000 goes into a one-year GIC at 10% per year, compounded semi-annually. *Calculate the balance after the year.*

80. Compounding frequency — A bank advertises 12% per year, compounded semi-annually. Before signing anything, a careful saver checks what that quote really pays over one full year. *Determine the effective annual rate, as a percent.*

81. The rule of 72 — A dividend portfolio grows at roughly 9% per year. *Estimate how long the money takes to double.*

82. The rule of 72 — A government bond yields 2% per year. *Estimate how long the money takes to double.*

83. Present value — $4000 sits in an account growing at 10% per year, compounded annually. Its owner is saving toward a purchase 3 years away. *Calculate what the money will be worth in 3 years.*

84. Present value — $900 sits in an account growing at 10% per year, compounded annually. Its owner is saving toward a purchase 1 year away. *Calculate what the money will be worth in 1 year.*

85. Simple vs compound — Twins each deposit $1000. Ari chooses simple interest at 10% per year; Bea chooses 10% per year compounded annually. Both leave the money for 3 years. *Compare the two balances and determine compound's edge.*

86. Simple vs compound — Twins each deposit $3000. Ari chooses simple interest at 10% per year; Bea chooses 10% per year compounded annually. Both leave the money for 3 years. *Compare the two balances and determine compound's edge.*

87. The Savings Ledger — Opening day at the credit union: $1000 goes into an account paying a clean 10% per year, compounded annually. Keep the ledger for three years — work each line, and every balance feeds the next. *Determine the balance at the end of each year, then settle the old simple-versus-compound score.*

88. The Savings Ledger — Bonus mark, no pencil needed: the account pays 10% per year, and the teller wonders aloud how long a deposit takes to double there. *Estimate the doubling time with the rule of 72.*

Answer key

1. 5 $/kg
2. 4 $/kg
3. 160 g
4. 160 g
5. 500,000 cm
6. 350,000 cm
7. 140 $
8. 160 $
9. 67.8 $
10. 132 $
11. 25 %
12. 81 $
13. 30 $
14. 5 $/kg
15. 3 (no unit)
16. -0.5 (no unit)
17. 2 (no unit)
18. -2 (no unit)
19. 13 $
20. 3 $/km
21. -0.25 (no unit)
22. 4 h
23. 3 h
24. 80 km
25. 12 L/100 km
26. 90 $
27. 7 m
28. 10 m
29. 2 h
30. 8 L/100 km
31. 48 m^2
32. 40 m^2
33. 7 m
34. 42 m^2
35. 94.2 m
36. 37.68 m
37. 62.8 cm^2
38. 23.55 m
39. 6 m
40. 8 m
41. 36 m^2
42. 320 m^2
43. 70 m^2
44. 5 $\$/m^2$
45. 18 m^3
46. 125 m^3
47. 4.99993 m
48. 169.646 m^3
49. 56.5487 m^3
50. 12 m^3
51. 523.599 m^3
52. 314.159 m^2
53. 22 m^2
54. 54 m^2
55. 251.327 cm^2
56. 1,526.81 cm^2
57. 7 L
58. 25,132.7 L
59. 7 m
60. 4.99945 m
61. 90 π m^3
62. 120 π m^2
63. 17 m
64. 13 m
65. 34.641 m
66. 16 m
67. 36.8699 °
68. 36.8699 °
69. 34.641 m
70. 17.3205 m
71. 20 %
72. 2 (n : 1)
73. 15 m
74. 5 %
75. 18 $
76. 3 years
77. 2,662 $
78. 2,205 $
79. 2,205 $
80. 12.36 %
81. 8 years
82. 36 years
83. 5,324 $
84. 990 $
85. 1,300 $
86. 3,900 $
87. 1,100 $
88. 7.2 years