

Grade 11 Math: Functions & Applications

Study Guide

Formulas · worked examples · practice problems with the answer key

Built to Ontario's MCF3M. Also covers parts of BC Pre-calculus 11
and Pre-calculus 12, and Alberta Mathematics 20-1 and 30-1.



5 modules · 43 formulas · 67 practice problems



The Exam Room · formula.expert · metric edition 1
October 2026 · Mathieu Leclerc, with Claude

About this study guide

Who it's for. Students working through Grade 11 Math: Functions & Applications, and the teachers, tutors and parents printing handouts for them. Print it, copy it, staple it: it's free for classroom use, whole or by the page.

How to use it. Each module teaches its formulas in order (equation, picture, and what every symbol means), then closes with a practice set. Work the problems with a pencil; the answer key waits at the back so honest work gets an honest check. The two-page formula sheet up front is the exam-day companion, and every formula carries its number (Math: Functions & Applications 11-1 onward) so your written work can cite exactly which one you used.

Do it live. Every problem in here has a living version in the Exam Room at formula.expert/exams/grade-11-math-functions: new numbers every attempt, hints that pop down when you want them, stars and streaks when you don't need them. When a printed problem stumps you, the matching lesson will coach you through it, step by step, free and with no sign-up. Other courses' study guides live at formula.expert/study-guides.

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The formula sheet

Module 1 • Quadratic Functions

Math: Functions & Applications 11-1 Vertex x-Coordinate of a Parabola

$$h = -\frac{b}{2a}$$

The x-coordinate of a parabola's vertex, sitting midway between the two roots and marking the axis...

Math: Functions & Applications 11-2 Discriminant of a Quadratic

$$\Delta = b^2 - 4ac$$

The quantity $b^2 - 4ac$ under the square root, which reveals how many real roots a quadratic has...

Math: Functions & Applications 11-3 Vertex y-Coordinate of a Parabola

$$k = c - \frac{b^2}{4a}$$

The y-coordinate of a parabola's vertex: its minimum value when a is positive, and its maximum when...

Math: Functions & Applications 11-4 Completing the Square: the Constant Needed

$$k = \left(\frac{b}{2}\right)^2$$

The constant you add to $x^2 + bx$ to turn it into a perfect square trinomial, the pivotal step in...

Math: Functions & Applications 11-5 Quadratic Formula (Positive Root)

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

The larger of the two solutions of $ax^2 + bx + c = 0$, taking the plus branch of the square root in...

Math: Functions & Applications 11-6 Quadratic Formula (Negative Root)

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The smaller of the two solutions of $ax^2 + bx + c = 0$, taking the minus branch of the square root in...

Math: Functions & Applications 11-7 Sum of the Roots of a Quadratic

$$S = -\frac{b}{a}$$

Vieta's relation giving the sum of a quadratic's two roots directly from its coefficients, without...

Math: Functions & Applications 11-8 Product of the Roots of a Quadratic

$$P = \frac{c}{a}$$

Vieta's relation giving the product of a quadratic's two roots directly from its coefficients...

Math: Functions & Applications 11-9 Displacement (Uniform Acceleration)

$$d = v_0 t + \frac{1}{2} a t^2$$

Distance travelled under constant acceleration, starting from an initial velocity, over a time t .

Math: Functions & Applications 11-10 Projectile Maximum Height

$$H = \frac{v_0^2 \sin^2 \theta}{2g}$$

Peak height reached by a projectile launched at a given speed and angle above level ground,...

Math: Functions & Applications 11-11 Projectile Time of Flight

$$T = \frac{2v_0 \sin \theta}{g}$$

Total time a projectile stays airborne before returning to its launch height, set by launch speed...

Module 2 • Exponential Growth & Decay

Math: Functions & Applications 11-12 Exponential Growth

$$A = A_0(1 + r)^t$$

Amount after t periods of compound growth at rate r per period.

Math: Functions & Applications 11-13 Exponential Decay

$$A = A_0(1 - r)^t$$

Amount remaining after t periods of losing a fixed fraction r per period.

Math: Functions & Applications 11-14 Percent Change

$$c = \frac{x_{\text{new}} - x_{\text{old}}}{x_{\text{old}}}$$

Relative change from an old value to a new one, as a fraction of the old value.

Math: Functions & Applications 11-15 Declining-Balance Depreciation (Book Value)

$$B = C(1 - d)^k$$

Book value of an asset after k years when a fixed fraction d of the remaining value is written off...

Math: Functions & Applications 11-16 Half-Life Decay

$$N = N_0 \left(\frac{1}{2}\right)^{t/t_{1/2}}$$

Gives the quantity remaining after repeated halvings over an elapsed time measured in half-lives.

Math: Functions & Applications 11-17 Half-Life and Decay Constant

$$t_{1/2} = \frac{\ln 2}{\lambda}$$

Converts between a half-life and the exponential decay constant via the factor $\ln 2 = 0.6931471806$.

Math: Functions & Applications 11-18 Exponential Growth by Doubling Time

$$N = N_0 \cdot 2^{t/T}$$

Growth of a quantity that doubles every fixed interval T .

Math: Functions & Applications 11-19 Rule of 72 (Doubling Time)

$$n \approx \frac{0.72}{i}$$

Mental-arithmetic estimate of how many periods it takes money to double at rate i : with the rate...

Math: Functions & Applications 11-20 Solve an Exponential Equation for the Exponent

$$x = \frac{\ln(y/a)}{\ln b}$$

Solves $a \cdot b^x = y$ for the unknown exponent x using logarithms, the standard move behind every growth...

Math: Functions & Applications 11-21 Logarithm Change of Base

$$\log_b x = \frac{\ln x}{\ln b}$$

Value of a logarithm in any base, rewritten with natural logarithms so an ordinary calculator can...

Module 3 • Periodic Functions

Math: Functions & Applications 11-22 Period-Frequency Relation

$$T = \frac{1}{f}$$

Period and frequency are reciprocals: seconds per cycle versus cycles per second.

Math: Functions & Applications 11-23 Wave Speed ($v = f\lambda$)

$$v = f\lambda$$

The universal wave equation: a wave's speed equals its frequency times its wavelength.

Math: Functions & Applications 11-24 Simple Pendulum Period

$$T = 2\pi \sqrt{\frac{L}{g}}$$

Period of a simple pendulum swinging through small angles, with $g = 9.80665 \text{ m/s}^2$ (standard gravity).

Math: Functions & Applications 11-25 Speed in Circular Motion ($v = 2\pi r/T$)

$$v = \frac{2\pi r}{T}$$

Speed of an object in uniform circular motion: one circumference ($2\pi r$, with $\pi \approx 3.14159265$) per...

Math: Functions & Applications 11-26 SHM Displacement at Time t

$$x = A \cos(\omega t)$$

Displacement of a simple harmonic oscillator at time t , a cosine of amplitude A and angular...

Math: Functions & Applications 11-27 Period of a Spring-Mass Oscillator

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Period of a mass bouncing on a spring, set only by the mass and the spring stiffness.

Module 4 · Trigonometry Applications

Math: Functions & Applications 11-28 Right-Triangle Sine Ratio (SOH)

$$\sin \theta = \frac{o}{h}$$

Relates an acute angle of a right triangle to its opposite side and the hypotenuse.

Math: Functions & Applications 11-29 Right-Triangle Cosine Ratio (CAH)

$$\cos \theta = \frac{a}{h}$$

Relates an acute angle of a right triangle to its adjacent side and the hypotenuse.

Math: Functions & Applications 11-30 Right-Triangle Tangent Ratio (TOA)

$$\tan \theta = \frac{o}{a}$$

Relates an acute angle of a right triangle to the ratio of its opposite and adjacent legs.

Math: Functions & Applications 11-31 Pythagorean Theorem

$$a^2 + b^2 = c^2$$

Relates the three sides of a right triangle.

Math: Functions & Applications 11-32 Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

In any triangle, each side is proportional to the sine of its opposite angle.

Math: Functions & Applications 11-33 Law of Cosines

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Finds the third side of any triangle from two sides and their included angle, or the angle from all...

Math: Functions & Applications 11-34 Triangle Area (Two Sides and Included Angle)

$$A = \frac{1}{2} ab \sin C$$

Computes a triangle's area from two sides and the angle between them, with no height needed.

Module 5 · Sequences & Financial Math

Math: Functions & Applications 11-35 Arithmetic Sequence nth Term

$$a_n = a_1 + (n - 1)d$$

The nth term of a sequence that steps by a fixed common difference.

Math: Functions & Applications 11-36 Geometric Sequence nth Term

$$a_n = a_1 r^{n-1}$$

The nth term of a sequence that multiplies by a fixed ratio each step.

Math: Functions & Applications 11-37 Arithmetic Series Sum

$$S_n = \frac{n}{2} (2a_1 + (n - 1)d)$$

Sum of the first n terms of an arithmetic sequence.

Math: Functions & Applications 11-38 Geometric Series Sum

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

Sum of the first n terms of a geometric sequence.

Math: Functions & Applications 11-39 Simple Interest

$$I = Prt$$

Interest earned on a principal at a flat rate per period, without compounding.

Math: Functions & Applications 11-40 Compound Interest (Periodic)

$$A = P \left(1 + \frac{r}{n}\right)^{nt}$$

Amount after t periods when interest compounds n times per period at rate r .

Math: Functions & Applications 11-41 Present Value

$$PV = \frac{FV}{(1 + r)^t}$$

Today's value of a future amount, discounted at rate r per period.

Math: Functions & Applications 11-42 Future Value of an Annuity (Regular Deposits)

$$FV = D \frac{(1 + i)^n - 1}{i}$$

What an equal deposit made at the end of every period grows to after n periods at rate i : the...

Math: Functions & Applications 11-43 Loan Payment (Amortized Loan or Mortgage)

$$M = \frac{Pi}{1 - (1 + i)^{-n}}$$

Level payment that retires a loan of principal P in exactly n payments at periodic interest rate i :...

MODULE 1

Quadratic Functions

Reading the parabola · The vertex · Completing the square · The quadratic formula · Counting the roots · Sum and product of roots · The height of a throw · The Apex

1.1 Vertex x-Coordinate of a Parabola

(Math: Functions & Applications 11-1)
$$h = -\frac{b}{2a}$$

Where,

h = Vertex x-coordinate

a = Coefficient of x^2

b = Coefficient of x

Every parabola $y = ax^2 + bx + c$ is symmetric about a single vertical line, and that line runs through its turning point at $x = -b/(2a)$. The expression is just the quadratic formula with the square root deleted: the two roots sit at $-b/(2a)$ plus and minus $\sqrt{\Delta}/(2a)$, so their midpoint, the vertex, is the shared first term. The equation of the axis of symmetry is literally $x = h$. Worked example: $y = 2x^2 - 8x + 5$ turns at $h = 8/4 = 2$, and its roots $2 \pm \sqrt{6}/2$ straddle that value symmetrically.

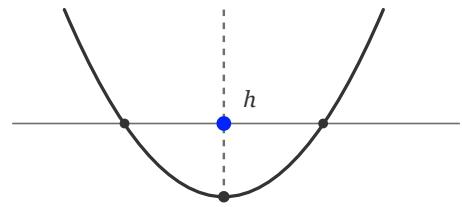


Figure 1.1: vertex x-coordinate of a parabola.

This is the workhorse of every optimisation question in a first algebra course, because a parabola's maximum or minimum can only occur at the vertex: fence 40 m of wire into the largest rectangle, and the area $A = x(20 - x) = -x^2 + 20x$ peaks at $h = -20/(2 \times -1) = 10$ m, a square. Descartes' *La Géométrie* of 1637 is what made this arithmetic possible at all: before coordinates, a parabola was a slice through a cone and its axis was found by construction, not by dividing two numbers. The trap is dropping the minus sign or forgetting the 2 in the denominator; if a is negative, remember that h can still be positive because two negatives cancel.

Worked example 1.1. Given $a = 2$, $b = -8$; find h (vertex x-coordinate). $y = 2x^2 - 8x + 5 \rightarrow \text{vertex at } x = 2$.

1.2 Discriminant of a Quadratic

(Math: Functions & Applications 11-2) $\Delta = b^2 - 4ac$

Where,

Δ = Discriminant

a = Coefficient of x^2

b = Coefficient of x

c = Constant term

The discriminant is the part of the quadratic formula that lives under the square root, and its sign alone answers "how many real solutions?" without solving anything. Positive means two distinct real roots (the parabola cuts the x-axis twice), zero means one repeated root (the parabola just kisses the axis at its vertex), and negative means no real roots at all (the parabola floats entirely above or below).
Worked example: $3x^2 - 4x - 2 = 0$ gives $\Delta = (-4)^2 - 4(3)(-2) = 16 + 24 = 40$, positive but not a perfect square, so two real irrational roots.

Worked example 1.2. Given $a = 3$, $b = -4$, $c = -2$; find Δ (discriminant). $3x^2 - 4x - 2 = 0 \rightarrow \Delta = 40$.

1.3 Vertex y-Coordinate of a Parabola

(Math: Functions & Applications 11-3) $k = c - \frac{b^2}{4a}$

Where,

k = Vertex y-coordinate

a = Coefficient of x^2

b = Coefficient of x

c = Constant term

Substituting the vertex abscissa $x = -b/(2a)$ back into $y = ax^2 + bx + c$ collapses to $k = c - b^2/(4a)$, the height of the turning point. Notice the discriminant hiding inside: $k = -\Delta/(4a)$, so a parabola with $\Delta = 0$ has its vertex exactly on the x-axis.
Worked example: $y = x^2 - 6x + 5$ has $k = 5 - 36/4 = -4$, so the vertex is $(3, -4)$ and the curve dips four units below the axis before climbing back through its roots at $x = 1$ and $x = 5$.

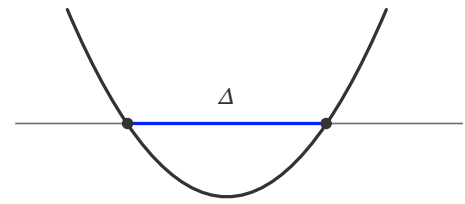


Figure 1.2: discriminant of a quadratic.

That last test is the practical one for students: Δ a perfect square (and a, b, c whole numbers) means the quadratic factors neatly over the integers, so you can skip the formula entirely. Engineers read the same quantity as a stability check: the discriminant of a system's characteristic equation decides whether a damped oscillator returns smoothly to rest or overshoots and rings. The common trap is sign bookkeeping: with c negative, $-4ac$ becomes an *addition*, which is why $\Delta = 40$ above rather than -8 . Square b first, then subtract, and keep the minus sign attached to c .

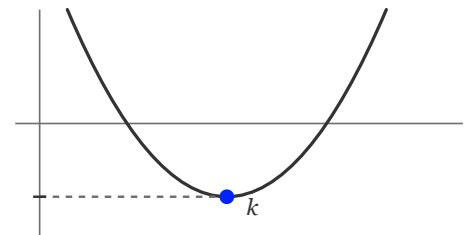


Figure 1.3: vertex y-coordinate of a parabola.

Because the vertex is the only turning point, k is the answer to "what is the smallest (or largest) value this expression can take?": a projectile launched at 20 m/s from ground level follows $h = 20t - 4.9t^2$, peaking at $k = 0 - 400/(4 \times -4.9) \approx 20.4$ m. Getting there by hand means completing the square, the same manoeuvre a Babylonian scribe used on tablet BM 13901 nearly four thousand years ago and that al-Khwārizmī later justified with a literal picture of a square with rectangles glued to two sides. The classic trap is sign

handling when a is negative: $4a$ is then negative, so subtracting $b^2/(4a)$ raises k , which is exactly right for a downward-opening curve.

Worked example 1.3. Given $a = 1$, $b = -6$, $c = 5$; find k (vertex y -coordinate). $y = x^2 - 6x + 5 \rightarrow \text{vertex } y = -4$.

1.4 Completing the Square: the Constant Needed

(Math: Functions & Applications 11-4) $k = \left(\frac{b}{2}\right)^2$

Where,

k = Constant to add

b = Coefficient of x

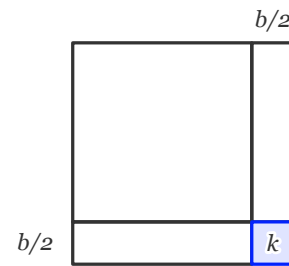


Figure 1.4: completing the square: the constant needed.

Halve the coefficient of x , square it, and add: that single instruction turns $x^2 + bx$ into the perfect square $(x + b/2)^2$.
Worked example: $x^2 + 10x$ needs $k = (10/2)^2 = 25$, giving $(x + 5)^2$. Once an expression is a perfect square you can take a square root of both sides and finish the problem in two lines, which is exactly how the quadratic formula itself is derived and how vertex form $a(x - h)^2 + k$ is reached.

The name is literal. A Babylonian scribe on tablet BM 13901, around 1800 BCE, solved "I added the area and the side of my square and got 45 (in base 60, i.e. $3/4$)" by halving the

side coefficient, squaring, and adjusting: arriving at $\frac{1}{2}$ with no algebraic symbols at all. Al-Khwārizmī in ninth-century Baghdad drew the same argument as an actual diagram: a square of side x with two rectangles of width $b/2$ glued to adjacent sides, leaving a missing corner of area $(b/2)^2$ to be filled in. Two traps. First, the leading coefficient must be 1 before you start: for $3x^2 + 12x$, factor out the 3 to get $3(x^2 + 4x)$ and complete the square inside. Second, adding k changes the expression, so in an equation you must add it to both sides, and in an expression you must immediately subtract it again to keep the value unchanged.

Worked example 1.4. Given $b = 10$; find k (constant to add). $x^2 + 10x \rightarrow \text{add } 25$.

1.5 Quadratic Formula (Positive Root)

(Math: Functions & Applications 11-5) $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$

Where,

x = Root (plus branch)

a = Coefficient of x^2

b = Coefficient of x

c = Constant term

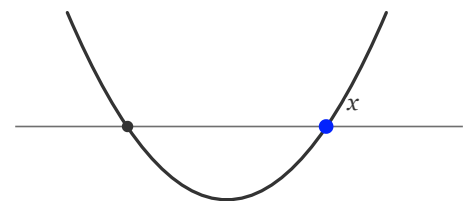


Figure 1.5: quadratic formula (positive root).

The quadratic formula solves *every* equation of the form $ax^2 + bx + c = 0$ in one stroke, and this page returns the plus branch: the root you get by *adding* the square root of the

discriminant. Its twin, the minus branch, is a separate calculator here because a solver returns one number at a time; run both with the same a , b and c to see the complete

pair. Worked example: $2x^2 + 3x - 5 = 0$ gives $b^2 - 4ac = 9 + 40 = 49$, so $x = (-3 + 7)/4 = 1$ here and $(-3 - 7)/4 = -2.5$ on the minus branch, and indeed $2(1)^2 + 3(1) - 5 = 0$.

The method is older than the notation. A Babylonian scribe working the clay tablet BM 13901 around 1800 BCE handled "I added the area and the side of my square and got 0;45" by halving the coefficient, squaring it, and adjusting: completing the square, three and a half millennia before

anyone wrote a formula down. Around 820 CE in Baghdad, al-Khwārizmī's *al-Kitāb al-mukhtaṣar fī ḥisāb al-jabr wa-l-muqābala* gave the word *al-jabr* ("restoration") to the whole subject, though with no negative numbers available he had to treat six separate cases. The trap to avoid: a must not be zero (the equation is then linear), and if the discriminant comes out negative there is no real root at all. The parabola never touches the x-axis. Watch the signs, too: with b negative, $-b$ is positive.

Worked example 1.5. Given $a = 2$, $b = 3$, $c = -5$; find x (root (plus branch)). $2x^2 + 3x - 5 = 0 \rightarrow$ plus root 1.

1.6 Quadratic Formula (Negative Root)

(Math: Functions & Applications 11-6)
$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Where,

x = Root (minus branch)

a = Coefficient of x^2

b = Coefficient of x

c = Constant term

This is the minus branch of the quadratic formula: the same machine as the plus branch, with the square root *subtracted* instead of added. Because a solver hands back one number, the two roots live on two pages here; feed identical a , b and c to both and you have the full solution set. Worked example: $2x^2 + 3x - 5 = 0$ has discriminant $9 + 40 = 49$, so this branch gives $x = (-3 - 7)/4 = -2.5$ while the plus branch gives 1. Check it: $2(-2.5)^2 + 3(-2.5) - 5 = 12.5 - 7.5 - 5 = 0$.

Worked example 1.6. Given $a = 2$, $b = 3$, $c = -5$; find x (root (minus branch)). $2x^2 + 3x - 5 = 0 \rightarrow$ minus root -2.5 .

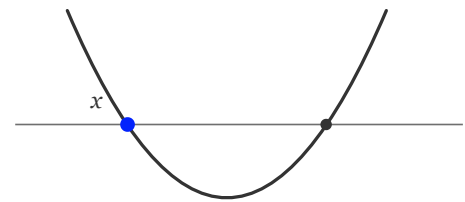


Figure 1.6: quadratic formula (negative root).

Two sanity checks catch most mistakes. The roots must sum to $-b/a$ (here $1 + (-2.5) = -1.5 = -3/2 \checkmark$) and multiply to c/a ($1 \times -2.5 = -2.5 = -5/2 \checkmark$): relations named for François Viète, who in 1591 became the first algebraist to use letters for the *known* coefficients as well as the unknowns, which is why we can write " $ax^2 + bx + c$ " at all. When a is positive the minus branch is the smaller root; when a is negative the labels swap, because dividing by a negative $2a$ flips the order. And if the discriminant is exactly zero, both branches return the same repeated root.

1.7 Sum of the Roots of a Quadratic

(Math: Functions & Applications 11-7)
$$S = -\frac{b}{a}$$

Where,

S = Sum of the roots

a = Coefficient of x^2

b = Coefficient of x

You can read the sum of a quadratic's two roots straight off its coefficients: expand $a(x - x_1)(x - x_2)$ and the x term comes out as $-a(x_1 + x_2)$, so $x_1 + x_2 = -b/a$. Worked example: $2x^2 - 7x + 3 = 0$ has $S = 7/2 = \mathbf{3.5}$, and the actual roots 3 and $\frac{1}{2}$ do indeed add to 3.5. No formula, no square roots, no arithmetic beyond one division.

The relation carries François Viète's name; his 1591 *In artem analyticam isagoge* introduced the habit of writing consonants for known coefficients and vowels for

Worked example 1.7. Given $a = 2$, $b = -7$; find S (sum of the roots). $2x^2 - 7x + 3 \rightarrow \text{root sum } 3.5$.

1.8 Product of the Roots of a Quadratic

(Math: Functions & Applications 11-8)
$$P = \frac{c}{a}$$

Where,

P = Product of the roots

a = Coefficient of x^2

c = Constant term

Multiply out $a(x - x_1)(x - x_2)$ and the constant term lands as $a \cdot x_1 x_2$, so the two roots always multiply to c/a . Worked example: $4x^2 + 5x - 9 = 0$ has $P = -9/4 = \mathbf{-2.25}$, and the roots 1 and $-9/4$ confirm it. Paired with the sum relation $S = -b/a$, this pins a quadratic completely: any equation whose roots add to S and multiply to P is $x^2 - Sx + P = 0$ up to an overall scale factor.

The sign of P is a free diagnostic. Negative means the roots straddle zero (one positive, one negative) so a projectile's height equation with a negative constant term always has

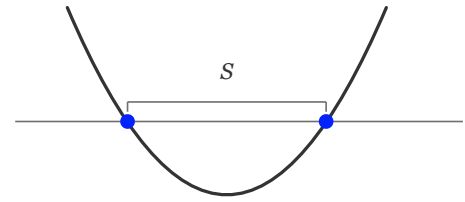


Figure 1.7: sum of the roots of a quadratic.

unknowns, and only with symbolic coefficients could such a statement even be phrased. Today it is the fastest way to check a solution, if your two roots do not sum to $-b/a$ you have made an arithmetic slip, and the fastest way to *build* a quadratic to order: roots summing to 5 and multiplying to 6 give $x^2 - 5x + 6 = 0$. Two traps: the minus sign is part of the formula ($S = -b/a$, not b/a), and it holds even when the roots are complex, where the imaginary parts cancel and the sum stays real.

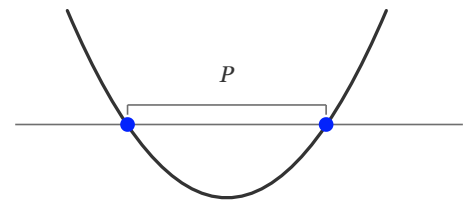


Figure 1.8: product of the roots of a quadratic.

one physically meaningless negative time. Positive P with positive S means both roots are positive; positive P with negative S means both are negative. The trap: $P = c/a$, with no minus sign, unlike the sum. And remember these relations describe the roots you would *get*, real or complex: a quadratic with no real roots still has a perfectly real product c/a , because the two complex conjugates multiply to the square of their modulus.

Worked example 1.8. Given $a = 4$, $c = -9$; find P (product of the roots). $4x^2 + 5x - 9 \rightarrow$ root product -2.25 .

1.9 Displacement (Uniform Acceleration)

(Math: Functions & Applications 11-9) $d = v_0 t + \frac{1}{2} a t^2$

Where,

d = Displacement (m)

v_0 = Initial velocity (m/s)

a = Acceleration (m/s²)

t = Time (s)

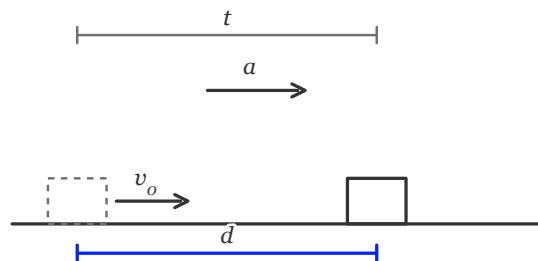


Figure 1.9: displacement (uniform acceleration).

When acceleration is constant, displacement has two parts: the distance you would cover at your initial velocity alone, plus the extra distance contributed by speeding up, and that extra grows with the *square* of time. A jet starting its takeoff roll from rest and holding 2 m/s^2 covers $d = 0 + \frac{1}{2}(2)(30^2) = 900 \text{ m}$ in 30 seconds, which is why runways are measured in kilometres.

The $\frac{1}{2}$ appears because the acceleration term is built from the average of a speed that grows linearly from zero. Galileo uncovered the underlying pattern (distances in successive equal time intervals follow the odd numbers 1, 3, 5, 7) by rolling bronze balls down inclined planes. Note that solving for t would mean solving a quadratic with potentially two positive roots, so this calculator rearranges only for d , v_0 , and a , where the answer is always single-valued.

Worked example 1.9. Given $v_0 = 0 \text{ m/s}$, $a = 2.5 \text{ m/s}^2$, $t = 30 \text{ s}$; find d (displacement). *Plane from rest, 2.5 m/s^2 for $30 \text{ s} \rightarrow 1125 \text{ m}$.*

1.10 Projectile Maximum Height

(Math: Functions & Applications 11-10) $H = \frac{v_0^2 \sin^2 \theta}{2g}$

Where,

H = Maximum height (m)

v_0 = Launch speed (m/s)

θ = Launch angle ($^\circ$)

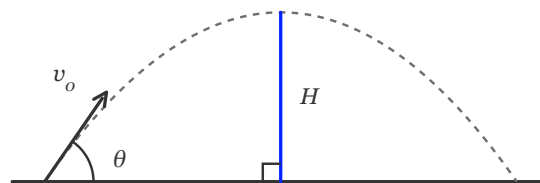


Figure 1.10: projectile maximum height.

Only the vertical slice of the launch velocity, $v_0 \sin \theta$, fights gravity, and it is spent entirely by the time the projectile stops climbing, so the apex sits at $v_0^2 \sin^2 \theta / 2g$. Fire straight up at 30 m/s and you reach $30^2 / (2 \times 9.80665) \approx 45.9 \text{ m}$, about a 15-storey building. Galileo established the underlying parabola in *Two New Sciences* (1638) by rolling inked bronze balls off a table and marking where they struck: the first clean demonstration that horizontal and vertical motion proceed independently.

The common trap is squaring the sine rather than the whole term, or confusing this with the range formula's $\sin 2\theta$. Note how much steeper the sensitivity to angle is here than for range: a 60° launch reaches three times the height of a 30° launch at the same speed, yet lands at exactly the same spot. Basketball players exploit that trade: a high arc reaches the rim moving slowly and nearly straight down, which makes the hoop look bigger.

Worked example 1.10. Given $v_0 = 30$ m/s, $\theta = 90^\circ$; find H (maximum height). *30 m/s straight up $\rightarrow$ 45.887 m.*

1.11 Projectile Time of Flight

(Math: Functions & Applications 11-11)
$$T = \frac{2v_0 \sin \theta}{g}$$

Where,

T = Time of flight (s)

v_0 = Launch speed (m/s)

θ = Launch angle ($^\circ$)

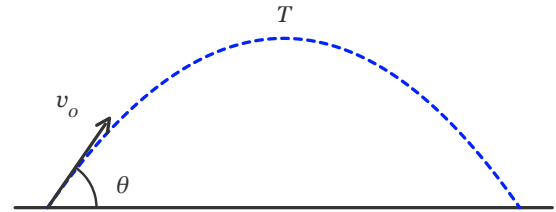


Figure 1.11: projectile time of flight.

Hang time depends only on the vertical launch velocity: the projectile takes $v_0 \sin \theta / g$ to reach the top and exactly as long to fall back, giving $T = 2v_0 \sin \theta / g$. Launch at 25 m/s and 30° and the vertical component is 12.5 m/s, so the flight lasts $2 \times 12.5 / 9.80665 \approx 2.55$ s. Nothing about the horizontal speed appears, which is Galileo's independence principle in its most testable form, the same insight behind his claim that a stone dropped from a moving ship's mast lands at the mast's foot.

Two cautions. First, this assumes the landing height equals the launch height; throw off a cliff or off a 2 m-tall shoulder and the real flight is longer, requiring the full quadratic. Second, the formula is symmetric, so a given time of flight matches two speeds only through the angle: solving for θ returns the principal arcsin branch (0° to 90°), and the supplementary angle $180^\circ - \theta$ gives the identical hang time on a lower, faster trajectory.

Worked example 1.11. Given $v_0 = 25$ m/s, $\theta = 30^\circ$; find T (time of flight). *25 m/s at $30^\circ \rightarrow 2.549$ s.*

Module 1 practice

1. The vertex: A design class models an arch with $y = x^2 + 6x + 10$. *Determine the x-coordinate of the vertex.*

2. The vertex: The graph of $y = 2x^2 + 4x + 5$ turns exactly once. *Determine the y-coordinate of the vertex.*

3. Completing the square: The expression $x^2 - 6x$ is one constant short of a perfect square trinomial. *Determine the constant that completes the square.*

4. Completing the square: Completing the square on $y = x^2 - 8x + 5$ produces a perfect square plus a leftover constant. *Determine the x-coordinate of the vertex the finished square reveals.*

5. The quadratic formula: Factoring stalls on the equation $x^2 + 1x - 12 = 0$, so the formula comes out. *Calculate the larger root.*

6. The quadratic formula: Factoring stalls on the equation $x^2 - 4x - 12 = 0$, so the formula comes out. *Calculate the larger root.*

7. Counting the roots: Before anyone solves $4x^2 + 5x + 4 = 0$, the exam asks how many real roots it has. *Calculate the discriminant, then count the crossings.*

8. Counting the roots: Before anyone solves $x^2 - 4x + 4 = 0$, the exam asks how many real roots it has. *Calculate the discriminant, then count the crossings.*

9. Sum and product of roots: The equation $x^2 - 6x + 8 = 0$ has two real roots, and Vieta reads them off the coefficients without solving a thing. *Determine what the two roots must add to.*

10. Sum and product of roots: The equation $2x^2 - 11x + 5 = 0$ has two real roots, and Vieta reads them off the coefficients without solving a thing. *Determine what the two roots must add to.*

11. The height of a throw: A ball leaves the hand moving straight up at 16 m/s. Take $g = 9.8 \text{ m/s}^2$. *Calculate the ball's height 1.0 s after launch.*

12. The height of a throw: A volleyball is served straight up at 22.4 m/s ($g = 9.8 \text{ m/s}^2$). *Determine the maximum height it reaches above the launch point.*

13. The Apex: Championship point. A ball is struck straight up at 20 m/s, and today $g = 10 \text{ m/s}^2$, stated on the paper — so its height follows $h(t) = 20t - 5t^2$. Work each line; every answer feeds the next. *Determine the apex time, the maximum height, the hang time — then make Vieta sign off.*

14. The Apex: Bonus mark, worked backwards: the other team's toss hangs in the air for exactly 2.0 s before it is struck ($g = 10 \text{ m/s}^2$). *Determine the speed it left the hand with.*

MODULE 2

Exponential Growth & Decay

Growth or decay · Percent change · Growing by a ratio · Decay and depreciation · Half-life · Doubling time · Solving for time · The Petri Dish

2.1 Exponential Growth

(Math: Functions & Applications 11-12) $A = A_0(1 + r)^t$

Where,

A = Final amount

A_0 = Initial amount

r = Growth rate per period (decimal)

t = Number of periods

Anything that grows by a fixed *percentage* each period (money at compound interest, populations, subscriber counts) multiplies by $(1 + r)$ every step, so growth compounds on itself. Enter r as a decimal: 5% is 0.05. Worked example: \$1,000 invested at 5% per year for 10 years gives $A = 1000 \times 1.05^{10} \approx \$1,628.89$. The extra \$128.89 beyond simple interest is interest earned on earlier interest. Einstein may never have called compound interest the eighth wonder of the world, but bankers behave as if he did.

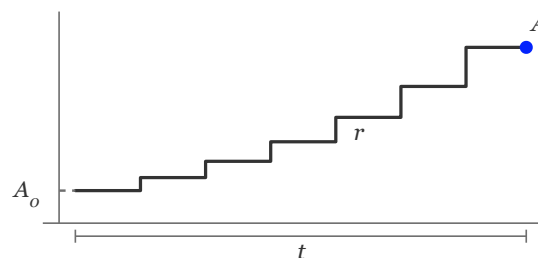


Figure 2.1: exponential growth.

The same formula runs backward. Solving for r extracts an average per-period growth rate from two snapshots, and solving for t (using logarithms) answers "how long until we reach A ?" A handy shortcut, the **Rule of 72**, falls straight out of the t rearrangement: dividing 72 by the percentage rate approximates the doubling time, so money at 6% doubles in roughly 12 years.

Worked example 2.1. Given $A_0 = 1000$, $r = 0.05$, $t = 10$; find A (final amount). *1000 at 5% for 10 periods $\rightarrow$ 1628.8946.*

2.2 Exponential Decay

(Math: Functions & Applications 11-13) $A = A_0(1 - r)^t$

Where,

A = Final amount

A_0 = Initial amount

r = Decay rate per period (decimal)

t = Number of periods

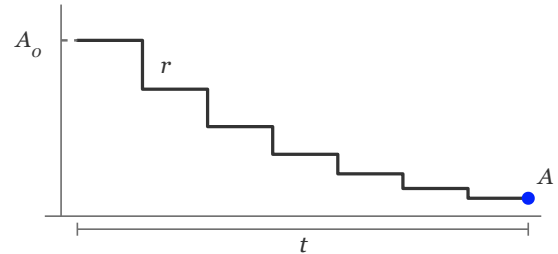


Figure 2.2: exponential decay.

Exponential decay is what happens when a quantity loses a fixed **percentage**, not a fixed amount, each period. Losing 20% means keeping 80%, so each step multiplies by $1 - r = 0.8$ and $A = A_0(1 - r)^t$. The consequence that separates this from linear loss is that the amount lost shrinks along with the amount remaining: the first year of that 20% takes \$6,000 off a \$30,000 car, the fifth year takes only \$2,458. The curve steepens nowhere and flattens forever, approaching zero without ever arriving.

A worked instance: that car, at 20% a year, is worth $30000 \times 0.8^5 = \$9,830$ after five years. The same shape governs coffee cooling toward room temperature, a drug clearing between doses, and chlorine residual fading in a treated tank: in each case something is leaving at a rate proportional to how much is still there, which is the physical condition that produces this curve.

The natural way to describe such a process is by its *half-life*, the time to fall to 50%. Setting $A/A_0 = 0.5$ and solving gives $t_{1/2} = \ln(0.5)/\ln(1 - r)$, so 20% a year corresponds to a half-life of 3.1 years. For small rates there

is a serviceable shortcut mirroring the Rule of 72: divide 70 by the percentage rate. The solver's rearrangements do the general version of this: solving for t answers "how long until it reaches this level?", and solving for r extracts the per-period rate from two measurements, which is how you find the decay constant of a real system rather than assuming one.

Three errors, and the third is the one that matters. First, r is a decimal: 20% is 0.20, and entering 20 makes $1 - r = -19$, producing an answer that flips sign every period and is obviously wrong if you look at it. Second, the rate and the time must share a clock: 20% per year with t in months is off by a factor no amount of care elsewhere will recover. Third, and worth stating flatly: **percentages do not add**. Losing 20% five times does not lose 100%; it leaves 32.8%. And a 20% drop followed by a 20% rise does not return you to where you started: it leaves 96%, because the rise is calculated on the smaller base. That asymmetry is why an investment that falls 50% needs a 100% gain to recover, and it catches people in every field where percentages get quoted casually.

Worked example 2.2. Given $A_0 = 20000$, $r = 0.15$, $t = 5$; find A (final amount). *20000 at 15% loss for 5 periods $\rightarrow$ 8874.10625.*

2.3 Percent Change

(Math: Functions & Applications 11-14)
$$c = \frac{x_{\text{new}} - x_{\text{old}}}{x_{\text{old}}}$$

Where,

c = Relative change

x_{new} = New value

x_{old} = Old value

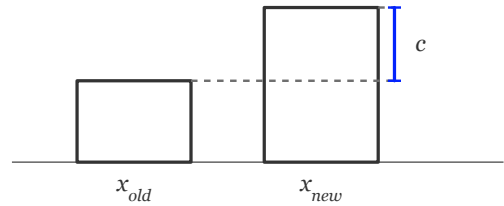


Figure 2.3: percent change.

Percent change measures a difference relative to where you started: a price moving from \$80 to \$92 changed by $(92 - 80)/80 = 0.15$, a 15% rise; a fall from \$80 to \$68 gives -0.15 . Pick the % display unit to see it as a percentage directly, and note that the old value is always the reference, which is why a 50% loss needs a 100% gain to break even.

The rearrangements handle both everyday directions: what a \$60 jacket costs after a 30% markup ($60 \times 1.30 = \$78$), and the reverse (the pre-sale price of an item now \$45 after a 25% discount, $45 / 0.75 = \$60$, not \$56.25 as adding 25% back would wrongly suggest).

Worked example 2.3. Given $x_{\text{new}} = 100$, $x_{\text{old}} = 80$; find c (relative change). $80 \rightarrow 100$ is a +25% change ($c = 0.25$).

2.4 Declining-Balance Depreciation (Book Value)

(Math: Functions & Applications 11-15)
$$B = C(1 - d)^k$$

Where,

B = Book value after k years (\$)

C = Initial cost (\$)

d = Depreciation rate per year (/yr)

k = Years elapsed

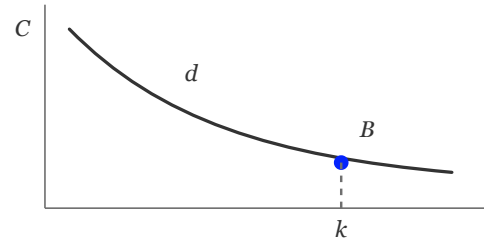


Figure 2.4: declining-balance depreciation (book value).

Declining balance takes the same fraction off whatever is left each year, so the write-down is large at first and shrinks steadily after. A \$20,000 asset at 20% a year goes 20,000, then 16,000, then 12,800, then 10,240, which the closed form $B = C(1 - d)^k$ reproduces as 20000×0.8^3 . This matches how equipment actually loses value far better than a straight line does, which is why most tax authorities use it. Canada's Capital Cost Allowance system is declining balance throughout.

There is a curiosity built into the mathematics: multiplying by a fraction repeatedly never reaches zero. In theory the asset is worth something forever. Accounting rules paper over this by switching to straight line near the end, or by writing off the remainder once it drops below a threshold. It also means you cannot solve for the rate from a book value of exactly zero, and the solver will tell you so.

The rate is often set from the life rather than measured. "Double declining balance" uses $d = 2/n$, so a ten-year asset is written down 20% a year, which is where the figure in the example above comes from.

Worked example 2.4. Given $C = 20000$ \$, $d = 20\%$, $k = 3$; find B (book value after k years). *\$20,000 at 20% declining, 3 years $\rightarrow$ \$10,240 book value.*

2.5 Half-Life Decay

(Math: Functions & Applications 11-16)
$$N = N_0 \left(\frac{1}{2} \right)^{t/t_{1/2}}$$

Where,

N = Remaining quantity

N_0 = Initial quantity

t = Elapsed time (s)

$t_{1/2}$ = Half-life (s)

Radioactive decay never runs out of sample all at once: it halves, and halves again, forever. Each half-life leaves exactly 50% of what entered it, so after two half-lives a quarter remains, after ten about a thousandth. Carbon-14, with its 5,730-year half-life, is the famous example: a bone whose C-14 content has fallen to one quarter of the living value has been dead roughly 11,460 years. N and N_0 can be in any matching unit (grams, atom counts, or becquerels) since only their ratio matters.

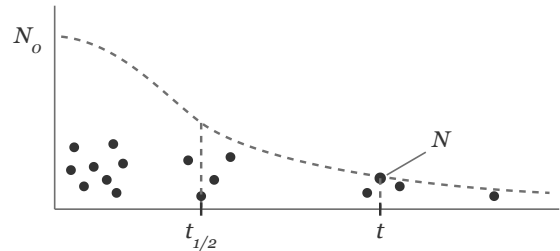


Figure 2.5: half-life decay.

Solving for time or half-life inverts the exponential with a base-2 logarithm, which demands $0 < N < N_0$: you cannot take the log of a zero ratio, and decay never leaves more than it started with. The same mathematics governs drug elimination in pharmacology and the six-hour half-life of technetium-99m, the workhorse isotope of medical imaging.

Worked example 2.5. Given $N_0 = 100$, $t = 11460$ yr, $t_{1/2} = 5730$ yr; find N (remaining quantity). *C-14: 100 units after 11460 yr (2 half-lives) $\rightarrow$ 25 remain.*

2.6 Half-Life and Decay Constant

(Math: Functions & Applications 11-17)
$$t_{1/2} = \frac{\ln 2}{\lambda}$$

Where,

$t_{1/2}$ = Half-life (s)

λ = Decay constant (Hz)

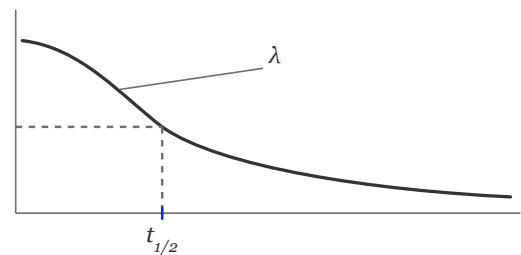


Figure 2.6: half-life and decay constant.

Half-life and decay constant are the same fact in two dialects. The decay constant λ is the fundamental one: it is the probability per unit time that any individual nucleus, chosen at random and regardless of its history, will decay. That single assumption, a constant per-nucleus probability, produces the exponential law $N = N_0 e^{-\lambda t}$ and nothing

else. The half-life is what you get by asking that law when $N/N_0 = \frac{1}{2}$: take logarithms and $t_{1/2} = \ln 2 / \lambda$, with $\ln 2 = 0.6931471806$. Large λ , short half-life. The two numbers carry identical information, and which one a source quotes is a matter of discipline rather than physics.

Cobalt-60 has a half-life of 5.27 years, so $\lambda = 0.6931/5.27 = 0.1315$ per year, or 4.17×10^{-9} per second once the years are converted. That per-second figure is the useful one, because activity is $A = \lambda N$. One gram of Co-60 holds $6.022 \times 10^{23}/60 = 1.00 \times 10^{22}$ nuclei, so its activity is $4.17 \times 10^{-9} \times 1.00 \times 10^{22} \approx 4.2 \times 10^{13}$ Bq: about **42 TBq, or 1100 curies, from a single gram**. λ is what converts a count of atoms into a radiation hazard.

A third quantity lives in the same family and is the one most often mixed up with the others. The mean lifetime τ , the average time an individual nucleus survives before decaying, is simply $1/\lambda$, which makes it $t_{1/2}/\ln 2 = 1.443 t_{1/2}$. **The mean life is 44% longer than the half-life, and they are not interchangeable.** Particle physics tables usually quote τ , nuclear medicine and health physics usually quote $t_{1/2}$, and a value copied across that boundary without conversion is wrong by a factor you will not notice from the magnitude alone. The same structure appears wherever a fixed fractional loss per unit time does: drug clearance, capacitor discharge through a resistor, the attenuation of light through an absorbing medium.

The unit trap is mechanical and unforgiving: λ and $t_{1/2}$ are reciprocals, so their units must be reciprocals too. A λ quoted per year set against a time measured in seconds is

off by 3.16×10^7 . Decide on one time unit at the start and hold it through the whole problem.

One property of λ is worth stating because it is genuinely unusual. It does not depend on temperature, on pressure, on the chemical compound the atom sits in, or on how long the nucleus has already existed. Essentially every chemical rate constant doubles or better for a 10 K rise; λ does not move. Heating a radioactive sample, dissolving it, or bonding it into a molecule changes nothing measurable, because the decay is a nuclear event and chemistry happens far outside the nucleus. In living systems, though, a second clock runs alongside: a radiopharmaceutical is also excreted, and the two combine as $1/T_{eff} = 1/T_{phys} + 1/T_{bio}$, so the effective half-life in a patient is always shorter than the physical one.

Worked example 2.6. Given $t_{1/2} = 138.4$ d; find λ (decay constant). *Po-210*: $t_{half} = 138.4$ d $\rightarrow \lambda = 5.79663e-8$ /s.

2.7 Exponential Growth by Doubling Time

(Math: Functions & Applications 11-18) $N = N_0 \cdot 2^{t/T}$

Where,

N = Final amount

N_0 = Initial amount

t = Elapsed time (s)

T = Doubling time (s)

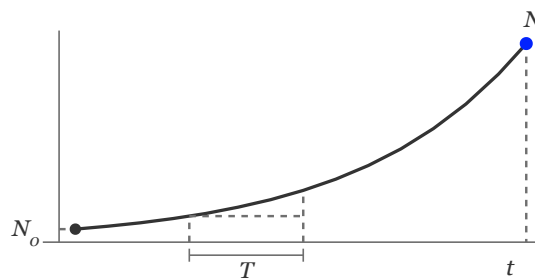


Figure 2.7: exponential growth by doubling time.

This is exponential growth written in the parameter people actually measure. Instead of a percentage per period, it uses T , the time to double, and the exponent t/T then simply counts how many doublings have gone by: $N = N_0 \cdot 2^{t/T}$. Three doublings is eight times, ten doublings is a bit over a thousand times, twenty is a million. Reading the exponent as a count of doublings makes the arithmetic something you can do in your head, which the percentage form never quite allows.

A worked instance. *E. coli* in favourable conditions divides about every 20 minutes. Left alone for eight hours that is 24 doublings, so a single cell becomes $2^{24} \approx 16.8$ million, which is the reason a water sample must be refrigerated and processed within hours, and why a warm sample line invalidates a count. Moore's law is the same equation at the other end of the time scale, with T around two years.

The bridge to a percentage rate is the Rule of 70. Since $T = \ln 2 / \ln(1 + r) \approx 0.693/r$ for small r , dividing 70 by the percentage growth per period gives the doubling time closely enough for mental arithmetic: 3.5% a year doubles

in about twenty years, 7% in about ten. Bankers use 72 instead of 70 because it divides more conveniently and is slightly more accurate over the range of interest rates they care about. Note also that only the *ratio* t/T enters, so the answer is unitless in time: any clock works, provided both use the same one.

That proviso is the first mistake: t in hours with T in minutes gives an exponent sixty times too large, and the result will be absurd rather than merely wrong, which at least makes it catchable. The second is a failure of intuition rather than arithmetic. Exponential growth does not feel fast until suddenly it is overwhelming, because **each doubling adds more than everything that came before it combined**. The pond that is fully covered by lilies on day thirty was half bare on day twenty-nine. The third is the honest limit: nothing doubles forever. Bacteria exhaust their substrate, markets saturate, transistors hit atomic scales. Every exponential in the real world is the early portion of an S-curve, and this formula describes only the part before the ceiling shows up. Using it to project far ahead is not a calculation so much as an assumption that nothing will run out.

Worked example 2.7. Given $N_0 = 1000$, $t = 12$ h, $T = 3$ h; find N (final amount). *1000 cells, $T = 3$ h, after 12 h $\rightarrow 16000$.*

2.8 Rule of 72 (Doubling Time)

(Math: Functions & Applications 11-19)
$$n \approx \frac{0.72}{i}$$

Where,

n = Periods to double

i = Rate per period

Divide 72 by the percentage rate and you have the years to double: 6% doubles in about twelve years, 9% in eight, 12% in six. It is the most useful piece of mental arithmetic in personal finance, and it has been in print since Luca Pacioli's *Summa de Arithmetica* of 1494, stated without proof as something merchants already knew.

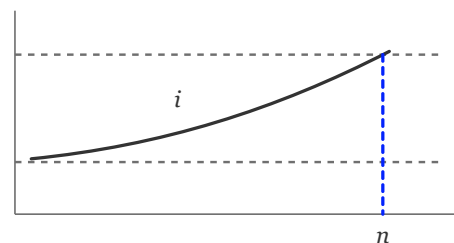


Figure 2.8: rule of 72 (doubling time).

The exact constant is not 72. Doubling requires $n \ln(1 + i) = \ln 2$, and for small rates that is close to $0.693/i$, so 69.3 would be more accurate, and exactly right for continuous compounding. Seventy-two is used because it divides cleanly by 2, 3, 4, 6, 8, 9 and 12, and because the small upward fudge happens to compensate for the

approximation across the range of rates people actually meet. It is at its best between about 6% and 10% and drifts noticeably above 20%.

The rule cuts both ways, which is the part worth remembering. At 3% inflation, prices double in 24 years, so a fixed pension halves in purchasing power over an ordinary retirement. The same arithmetic that makes savings look encouraging makes inflation look alarming.

Worked example 2.8. Given $i = 6\%$; find n (periods to double). *6% a year $\rightarrow$ doubles in about 12 years.*

2.9 Solve an Exponential Equation for the Exponent

(Math: Functions & Applications 11-20)
$$x = \frac{\ln(y/a)}{\ln b}$$

Where,

x = Unknown exponent

a = Coefficient

b = Base

y = Result

An unknown stuck in an exponent cannot be reached by dividing or subtracting; you have to take logarithms of both sides. Divide off the coefficient first, then apply the power rule: $a \cdot b^x = y$ becomes $b^x = y/a$, and $x = \ln(y/a) / \ln b$. Worked example: $5 \cdot 2^x = 320$ gives $2^x = 64$, so $x = \ln 64 / \ln 2 = 6$. The natural log is only a convenience: base 10 gives the identical answer, because the two conversion factors cancel.

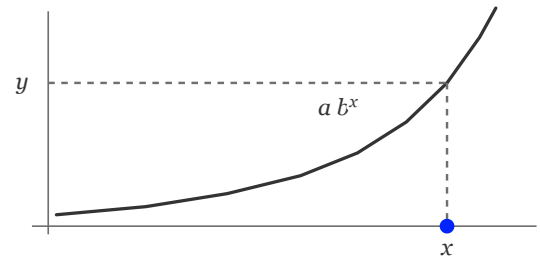


Figure 2.9: solve an exponential equation for the exponent.

This one manoeuvre answers a whole family of questions that otherwise need guesswork: how many years until an investment reaches a target, how long until a drug concentration falls below a threshold, how many half-lives a sample has been through. Napier built his tables in 1614 for precisely this reason, and the Rule of 72 that bankers still quote is nothing more than $\ln 2 \approx 0.693$ rounded up and dressed for mental arithmetic. Traps to watch: divide by the coefficient *before* logging, since $\ln(5 \cdot 2^x)$ is $\ln 5 + x \ln 2$, not $5x \ln 2$; and a base of exactly 1 makes the equation unsolvable, because 1^x never budges from 1.

Worked example 2.9. Given $a = 5$, $b = 2$, $y = 320$; find x (unknown exponent). $5 \cdot 2^x = 320 \rightarrow x = 6$.

2.10 Logarithm Change of Base

(Math: Functions & Applications 11-21)
$$\log_b x = \frac{\ln x}{\ln b}$$

Where,

L = Logarithm value

x = Argument

b = Base

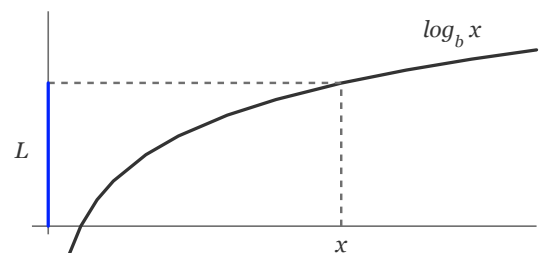


Figure 2.10: logarithm change of base.

Calculators carry two logarithm buttons, base 10 and base e , but questions arrive in every base, so change of base rewrites $\log_b x$ as a ratio of logs you can actually press. Any common base works, because the conversion factors cancel: $\log_2 32 = \ln 32 / \ln 2 = \log_{10} 32 / \log_{10} 2 = 5$ either way. Solving in the other direction identifies an unknown base: if $\log_b 1000 = 3$, then $b = 1000^{1/3} = 10$.

Logarithms exist for exactly this reason: to trade hard arithmetic for easy arithmetic. John Napier published *Mirifici Logarithmorum Canonis Descriptio* in 1614 after twenty

years of hand calculation, and Laplace later said the invention, "by shortening the labours, doubled the life of the astronomer." Every slide rule, every log table, and every decibel, pH and Richter reading descends from it. The traps are domain traps: x must be strictly positive (no real power of a positive base ever gives zero or a negative), and the base must be positive and not 1. Note also which quantity goes where: $\log_b x$ is $\ln x$ over $\ln b$, argument on top; flipping them gives the reciprocal, so $\log_2 8 = 3$ becomes 0.333 if you invert.

Worked example 2.10. Given $x = 32$, $b = 2$; find L (logarithm value). $\log_2(32) \rightarrow 5$.

Module 2 practice

15. Percent change: A hardware store's cordless drill was priced at \$80 last season and is \$100 today. *Calculate the percent change in the price.*

16. Percent change: Attendance at a school's science fair moved from 500 visitors last year to 400 visitors this year. *Determine the percent change in attendance.*

17. Growing by a ratio: A biologist counts 1600 beetles in a monitored plot. The colony is modelled to grow 25% each year. *Determine the beetle count after 2 years.*

18. Growing by a ratio: A monitored beetle colony grew from 1600 to 2500 over 2 years, gaining the same percent each year. *Determine the annual growth rate, as a percent.*

19. Decay and depreciation: A 500 mg dose falls to 320 mg over 2 hours, with the body clearing the same percent of whatever remains each hour. *Determine the hourly clearance rate, as a percent.*

20. Decay and depreciation: A mini-excavator bought for \$20,000 carries a book value of \$10,240 after 3 years, having lost the same percent of its remaining value each year. *Determine the annual depreciation rate.*

21. Half-life: A patient is given a 240 mg dose of a medical tracer whose half-life is 5 hours. *Determine the mass of tracer remaining after 15 hours.*

22. Half-life: A sample of a radioisotope falls from 200 counts per minute to 50 counts per minute over 30 minutes. *Determine the isotope's half-life.*

23. Doubling time: A bacterial culture starts with 100 cells and doubles every 30 minutes under laboratory conditions. *Determine the number of cells present after 180 minutes.*

24. Doubling time: A yeast culture grows from 1,000 cells to 16,000 cells in 24 hours, doubling at a steady rate throughout. *Determine the culture's doubling time.*

25. Solving for time: An exam question ends at log base 2 of 256, and the calculator on the desk offers only $\ln$ and $\log$ base 10. *Determine the value of log base 2 of 256.*

26. Solving for time: An exam question ends at log base 3 of 81, and the calculator on the desk offers only $\ln$ and $\log$ base 10. *Determine the value of log base 3 of 81.*

27. The Petri Dish: Final plate of the practical. A culture starts at 300 cells and doubles every 20 minutes. A fluorescent marker dye is added at 48 μg , and the dye breaks down with a half-life of 40 minutes. The incubator runs for 80 minutes. Work each line — every answer feeds the next. *Determine the number of cells per microgram of dye at the end of the run.*

28. The Petri Dish: Bonus mark, and a decision. Jar A is seeded with 50 cells and doubles every 20 minutes. Jar B is seeded with 1000 cells and doubles every 60 minutes. Both jars go into the same incubator for 120 minutes. *Determine which jar holds more cells when the incubator opens.*

MODULE 3

Periodic Functions

Reading the cycle · Period and frequency · The wave equation · The pendulum · Around the wheel · A snapshot in time · The Metronome

3.1 Period-Frequency Relation

(Math: Functions & Applications 11-22)
$$T = \frac{1}{f}$$

Where,

T = Period (s)

f = Frequency (Hz)

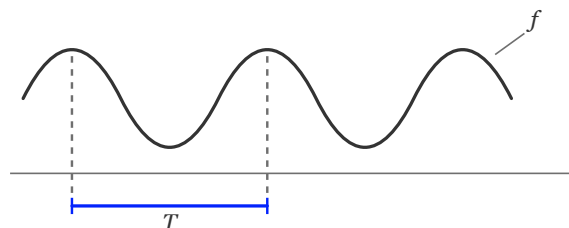


Figure 3.1: period-frequency relation.

Period and frequency are one fact counted in opposite directions. The period T is seconds per cycle; the frequency f is cycles per second; and $T = 1/f$ is not a discovery about nature but unit algebra. If something completes four cycles in a second, each cycle takes a quarter of a second, and no experiment was required to establish that. What earns this relation a page of its own is that almost every oscillation formula you will meet returns one member of the pair while the question in front of you wants the other, so this conversion sits quietly in the middle of nearly every wave calculation.

North American mains alternates at 60 Hz, so one full cycle takes $1/60 = 16.67$ ms: the interval behind the familiar hum in audio gear. European mains at 50 Hz gives 20 ms. Concert A at 440 Hz gives 2.27 ms per cycle. Run it the

other way and a resting heart beating once every 0.8 s is oscillating at 1.25 Hz, which is the same statement as 75 beats per minute.

The unit is younger than the idea. Frequency was written "cycles per second" well into the twentieth century; the International Electrotechnical Commission proposed **hertz** in 1930 and SI adopted it in 1960, honouring Heinrich Hertz, who between 1886 and 1888 generated and detected radio waves in his Karlsruhe laboratory and so turned Maxwell's equations from mathematics into an observed fact. The hertz is dimensionally just s^{-1} , and it is reserved by convention for periodic phenomena: the becquerel is also s^{-1} and counts random decays, which is precisely why the two units are kept apart despite being numerically identical. Combined with $v = f\lambda$, this page also gives the equally useful $v = \lambda/T$.

Three traps. The first is **angular** frequency: $\omega = 2\pi f$ in radians per second, and the pendulum and spring formulas carry their 2π for exactly this reason. Substituting an ω where an f belongs makes the answer wrong by a factor of 6.283, which is large enough to notice and small enough to rationalise. The second is revolutions per minute: 3600 rpm is 60 Hz, not 3600, and the conversion is a division by 60. The third catches people with pendulums: a "seconds pendulum" ticks once per second but has a period of *two* seconds, because a full cycle is out and back. Count a cycle as a return to the starting state moving in the starting direction, and the reciprocal will behave.

Worked example 3.1. Given $f = 60$ Hz; find T (period). *60 Hz mains $\rightarrow T = 1/60$ s.*

3.2 Wave Speed ($v = f\lambda$)

(Math: Functions & Applications 11-23) $v = f\lambda$

Where,

v = Wave speed (m/s)

f = Frequency (Hz)

λ = Wavelength (m)

Every traveling wave advances exactly one wavelength during each cycle of its source, and it completes f cycles every second, so its speed is simply frequency times wavelength. The relation holds for every wave in nature: sound, light, water ripples, seismic tremors. An FM station broadcasting at 100 MHz emits radio waves that travel at the speed of light, about 3.00×10^8 m/s, so each wave is roughly 3 m long, which is why FM antennas are built around three-quarters of a metre, a quarter of a wavelength.

Worked example 3.2. Given $v = 1234.8$ km/h, $f = 0.5$ kHz; find λ (wavelength). *Sound at 1234.8 km/h, 0.5 kHz $\rightarrow \lambda = 0.686$ m.*

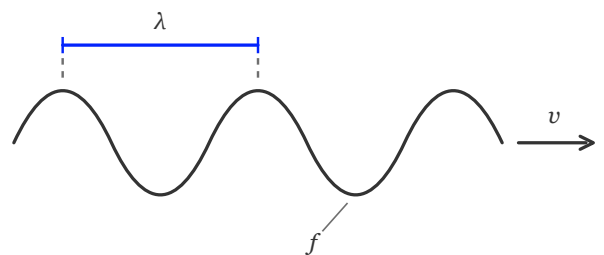


Figure 3.2: wave speed ($v = f\lambda$).

The common trap is thinking a higher frequency makes a wave *faster*. It doesn't: speed is set by the medium alone. Sound in room-temperature air moves at about 343 m/s whether it is a 20 Hz bass rumble ($\lambda \approx 17$ m) or a 20 kHz whistle ($\lambda \approx 17$ mm). Raise the frequency and the wavelength shrinks in exact proportion, leaving v untouched.

3.3 Simple Pendulum Period

(Math: Functions & Applications 11-24)
$$T = 2\pi\sqrt{\frac{L}{g}}$$

Where,

T = Period (s)

L = Pendulum length (m)

Galileo noticed, reputedly while timing a swinging cathedral lamp against his own pulse, that a pendulum's period depends on neither its mass nor, for small swings, its amplitude: only on its length and the local pull of gravity. Christiaan Huygens turned that insight into the pendulum clock in 1656, and for nearly three centuries it remained the world's best timekeeper. The solver uses standard gravity, $g = 9.80665 \text{ m/s}^2$, a value exact by definition.

Worked example 3.3. Given $L = 1 \text{ m}$; find T (period). $1 \text{ m pendulum} \rightarrow T = 2.006409 \text{ s}$.

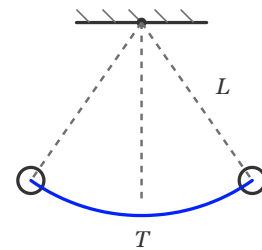


Figure 3.3: simple pendulum period.

A worked example: a "seconds pendulum" beating once per second has a full period of 2 s, so $L = g(T/2\pi)^2 = 9.80665 \times (2/6.2832)^2 \approx 0.994 \text{ m}$, the reason grandfather clocks stand about a metre tall inside. The formula is a small-angle approximation, accurate to about 1% for swings under 15° . Surveyors once ran it in reverse, timing precision pendulums to map tiny local variations in g across the Earth's surface.

3.4 Speed in Circular Motion ($v = 2\pi r/T$)

(Math: Functions & Applications 11-25)
$$v = \frac{2\pi r}{T}$$

Where,

v = Speed (m/s)

r = Radius (m)

T = Period (s)

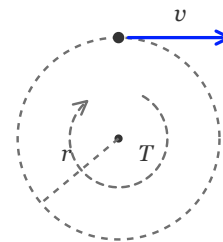


Figure 3.4: speed in circular motion ($v = 2\pi r/t$).

This is nothing more than speed equals distance over time, applied to a circle. One complete trip round covers a circumference of $2\pi r$, and it takes one period T , so the speed is $v = 2\pi r/T$. Everything difficult about circular motion lives in the direction of the velocity, which is changing constantly; the *magnitude* is this piece of grade-school arithmetic and nothing more.

The International Space Station orbits about 410 km above the surface, and Earth's mean radius is 6371 km, so its orbital radius is $r \approx 6781 \text{ km}$. It completes one orbit in roughly 93 minutes, which is 5580 s. That gives $v = 2\pi \times$

$6.781 \times 10^6 / 5580 \approx 7630 \text{ m/s}$, or 7.6 km/s: about 27 500 km/h, and the reason its crew see sixteen sunrises a day.

Combine it with the two neighbouring pages and a lot falls out. Since $\omega = 2\pi/T$, this equation is exactly $v = \omega r$ with the period substituted in. Put it into the centripetal acceleration $a_c = v^2/r$ and you get $a_c = 4\pi^2 r/T^2$; set that equal to the gravitational acceleration GM/r^2 and rearrange, and $T^2 \propto r^3$ drops out: Kepler's third law, derived in three lines from a circumference and Newton's law of gravitation.

The radius is measured from the centre of rotation, and for orbits that means from the centre of the Earth, not from the ground. Using the ISS's 410 km altitude as r instead of its 6781 km orbital radius understates the speed by a factor of about four and is the most common way this calculation goes wrong. The same principle applies on a smaller scale: for a point on a flywheel, r runs from the shaft axis, and for a car on a banked track it runs to the centre of the curve, not to the inside edge of the road. Two more. T is the time for one *full* revolution: a rotation rate given in rev/min has to be inverted first, $T = 60/N$ seconds, so 1800 rpm is a period of 0.0333 s, not 1800 of anything. And this describes *uniform* circular motion. A real planetary orbit is an ellipse on which the speed varies continuously, fastest at perihelion and slowest at aphelion; $2\pi r/T$ with a mean radius gives an average, not the speed at any particular moment.

Worked example 3.4. Given $r = 100$ m, $T = 20$ s; find v (speed). $r = 100$ m, $T = 20$ s $\rightarrow v = 10\pi \approx 31.4159$ m/s.

3.5 SHM Displacement at Time t

(Math: Functions & Applications 11-26) $x = A \cos(\omega t)$

Where,

x = Displacement (m)

A = Amplitude (m)

ω = Angular frequency (rad/s)

t = Time (s)

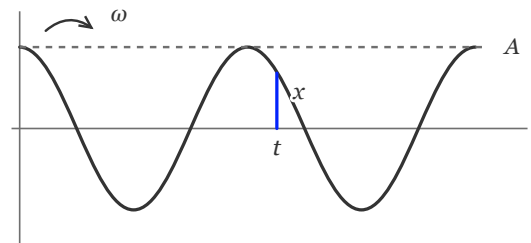


Figure 3.5: shm displacement at time t .

Any system whose restoring force grows in proportion to displacement (a mass on a spring, a small-swing pendulum, a quartz tuning fork) traces a cosine in time: $x = A \cos(\omega t)$, where $\omega = 2\pi f$ is the angular frequency in rad/s. Released from full stretch, a 5 cm amplitude oscillator at $\omega = 4$ rad/s sits at $0.05 \times \cos(2 \text{ rad}) \approx -0.0208$ m after half a second, already past centre and heading for the far side.

The cosine form assumes the clock starts at maximum displacement; start from the equilibrium point instead and you need a sine, or equivalently a phase shift. Christiaan Huygens turned this regularity into the first pendulum clock in 1656, cutting timekeeping error from about 15 minutes a day to roughly 15 seconds; today the same equation runs the 32,768 Hz quartz crystal in every wristwatch. Note that solving for t or ω returns only the first solution from arccos: the motion repeats forever, so infinitely many later times give the same displacement.

Worked example 3.5. Given $A = 0.05$ m, $\omega = 4$ rad/s, $t = 0.5$ s; find x (displacement). $A = 5$ cm, $\omega = 4$ rad/s, $t = 0.5$ s $\rightarrow -20.81$ mm.

3.6 Period of a Spring-Mass Oscillator

(Math: Functions & Applications 11-27) $T = 2\pi\sqrt{\frac{m}{k}}$

Where,

T = Period (s)

m = Mass (kg)

k = Spring constant (N/m)

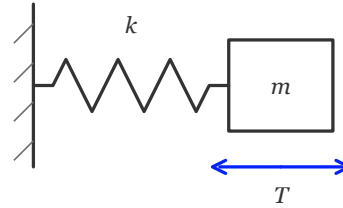


Figure 3.6: period of a spring-mass oscillator.

The remarkable claim in $T = 2\pi\sqrt{m/k}$ is what it leaves out: the amplitude. Pull the mass twice as far and it takes exactly as long to come back. The reason is the shape of the spring's restoring force, $F = -kx$: doubling the displacement doubles the distance to be covered, but it also doubles the force and therefore the acceleration, and the two effects cancel exactly. That cancellation is the definition of simple harmonic motion, and it is why springs and pendulums became clocks: a mechanism whose rate does not drift as its drive weakens is worth a great deal. The rest of the formula is intuition made quantitative. More mass means more inertia and a slower bounce; more stiffness means a firmer push back and a faster one. The square root softens both: quadrupling the mass only halves the frequency.

One kilogram on a 100 N/m spring gives $T = 2\pi\sqrt{1/100} = 0.628$ s, a frequency of 1.59 Hz. Run in reverse, the relation becomes a measurement. A 250 kg machine sitting on vibration isolators is observed to bounce at 3 Hz, so $T = 0.333$ s and $k = 4\pi^2m/T^2 = 4\pi^2(250)/0.111 \approx 89$ kN/m: the effective stiffness of the whole set of mounts, which is exactly the number you need before deciding whether the machine will resonate with anything the floor is doing.

Notice that g does not appear, which surprises people who have hung the mass vertically rather than laid it on a table. Gravity does change something: it shifts the equilibrium position down by the static deflection $\delta = mg/k$. But it shifts the whole motion, not its shape, so the oscillation about the new equilibrium runs at the same period as before. That gives a genuinely useful field trick: substitute $k = mg/\delta$ and the formula becomes $T = 2\pi\sqrt{\delta/g}$, so the natural frequency of a mounted machine can be read straight off how far it settled onto its mounts. A millimetre of static deflection means about 15.8 Hz; ten millimetres means 5 Hz. The same $2\pi\sqrt{\text{inertia}/\text{stiffness}}$ skeleton carries the pendulum, where L/g plays the part of m/k .

Three things to watch. The m is the oscillating mass, and a heavy spring oscillates too: adding about a third of the spring's own mass to m is the standard correction, and it matters on soft suspension springs. The k must be the effective stiffness of the entire arrangement, and springs combine the opposite way to resistors: parallel springs add directly, $k = k_1 + k_2$, while springs in series add reciprocally. Getting that backwards is common and the error is large. Finally, real springs are linear only over part of their travel; push into coil bind or use a progressive rate and the period becomes amplitude-dependent again, which means the answer this page gives quietly stops applying at exactly the point you were probably interested in.

Worked example 3.6. Given $m = 1$ kg, $k = 100$ N/m; find T (period). *1 kg on 100 N/m spring $\rightarrow T = 0.628319$ s.*

Module 3 practice

29. Period and frequency: A pendulum in the science hallway completes one full swing every 2 s. *Calculate the frequency of the swing.*

30. Period and frequency: A metronome on the piano clicks steadily at 2.5 Hz. *Calculate the time between one click and the next.*

31. The wave equation: In gym class, students shake one end of a long rope at 3 Hz, sending waves of wavelength 1.5 m down its length. *Calculate the speed of the waves along the rope.*

32. The wave equation: A skipping rope carries waves at 15 m/s while the turner's hand drives it at 3 Hz. *Determine the wavelength of the waves.*

33. The pendulum: A souvenir keychain pendulum hangs 0.25 m from pivot to bob. A student sets it swinging gently and times the swings. ($g = 9.8 \text{ m/s}^2$.) *Determine the period of one full swing.*

34. The pendulum: A gymnasium climbing rope hangs 3 m from its beam. A student sets it swinging gently and times the swings. ($g = 9.8 \text{ m/s}^2$.) *Determine the period of one full swing.*

35. Around the wheel: A Ferris wheel of radius 30 m turns smoothly, completing one revolution every 10 s. *Calculate the speed of a rider on the rim.*

36. Around the wheel: A Ferris wheel of radius 15 m turns smoothly, completing one revolution every 30 s. *Calculate the speed of a rider on the rim.*

37. A snapshot in time: A mass hangs from a spring. It is pulled 8 cm from its rest position and released; taking the release side as positive, it bobs with a period of 12 s. *Evaluate the displacement exactly 12 s after release.*

38. A snapshot in time: In the physics annex, a 4 kg lab cart sits between springs of combined stiffness 1 N/m. Nudged off centre, it oscillates freely back and forth. *Determine the period of the oscillation.*

39. The Metronome: Recital day. The practice-room metronome is set to 120 beats per minute. Sound crosses the room at 340 m/s today, and the metre-long pendulum on the wall clock swings once in exactly 2.0 s. Work each line — every answer feeds the next. *Determine the beat's period, the wavelength each click casts, and the pendulum that would keep this beat.*

40. The Metronome: Bonus mark, worked backwards: a tuning fork on the piano lid casts a sound wave 0.85 m long in the same 340 m/s air. *Determine the period of the fork's vibration, frequency first.*

MODULE 4

Trigonometry Applications

Naming the sides · Choosing the ratio · Heights and shadows · The sine law · The cosine law · Area without the height · The Survey

4.1 Right-Triangle Sine Ratio (SOH)

(Math: Functions & Applications 11-28) $\sin \theta = \frac{o}{h}$

Where,

θ = Acute angle ($^{\circ}$)

o = Opposite side (m)

h = Hypotenuse (m)

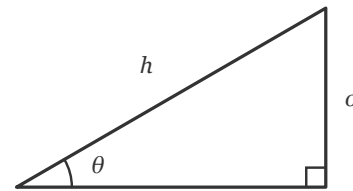


Figure 4.1: right-triangle sine ratio (soh).

SOH, **S**ine is **O**pposite over **H**ypotenuse, is the first line of the most durable mnemonic in mathematics. In a right triangle, the sine of an acute angle is fixed by the triangle's shape alone: every right triangle with a 30° angle has an opposite side exactly half its hypotenuse, no matter the size. That constancy is what makes the ratio a tool. A field example: a 20 m guy wire anchored at 30° to the ground reaches a height of $20 \times \sin 30^{\circ} = 10$ m up the mast. The

idea is ancient: Indian astronomers tabulated the *jya* (half-chord) around 500 CE, and a translation detour through Arabic and Latin gave us the word *sine*.

Solving for the angle uses the inverse function: $\theta = \arcsin(o/h)$. The solver returns the principal branch only, which here is exactly right: the non-right angles of a right triangle are always acute, so the answer between 0° and 90° is the only valid one. The ratio o/h must stay below 1, since a leg can never outgrow the hypotenuse.

Worked example 4.1. Given $\theta = 30^{\circ}$, $h = 10$ m; find o (opposite side). 30° angle, 10 m hypotenuse $\rightarrow$ 5 m opposite side.

4.2 Right-Triangle Cosine Ratio (CAH)

(Math: Functions & Applications 11-29) $\cos \theta = \frac{a}{h}$

Where,

θ = Acute angle ($^{\circ}$)

a = Adjacent side (m)

h = Hypotenuse (m)

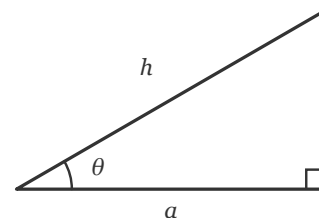


Figure 4.2: right-triangle cosine ratio (cah).

CAH: **C**osine is **A**djacent over **H**ypotenuse. Like all the trigonometric ratios, it works because the shape of a right triangle is fixed by one acute angle alone: every right triangle with a 60° angle has an adjacent leg exactly half its hypotenuse, whether it is drawn on a napkin or laid out across a field. Similar triangles keep the ratio constant, and that constancy is what turns an angle into a length. The name is a contraction of *complementi sinus*, the sine of the complement, because $\cos \theta = \sin(90^\circ - \theta)$: cosine is not a second idea but the same idea viewed from the other acute corner.

A worked instance you can check against a wall. A ladder is meant to stand at about 75° to the ground, the familiar one-out-for-four-up rule. A 6 m ladder therefore has its feet $6 \cos 75^\circ = 1.55$ m from the base of the wall, and reaches 5.8 m up it. Run the other way, $\theta = \arccos(a/h)$: a 6 m ladder set 2.0 m out is standing at $\arccos(0.333) = 70.5^\circ$, flatter than it should be.

Three values are worth knowing cold: $\cos 0^\circ = 1$, $\cos 60^\circ = 0.5$, $\cos 90^\circ = 0$. Beyond the triangle, cosine is the universal "how much of this points that way" operator.

Worked example 4.2. Given $\theta = 60^\circ$, $h = 8$ m; find a (adjacent side). *60° angle, 8 m hypotenuse $\rightarrow$ 4 m adjacent side.*

4.3 Right-Triangle Tangent Ratio (TOA)

(Math: Functions & Applications 11-30) $\tan \theta = \frac{o}{a}$

Where,

θ = Acute angle ($^\circ$)

o = Opposite side (m)

a = Adjacent side (m)

TOA: **T**angent is **O**pposite over **A**djacent. It is the ratio to reach for when the hypotenuse is unknown, and in the field the hypotenuse usually *is* unknown, because it runs along your line of sight to something you cannot put a tape on. Tangent is not independent of the other two: dividing

The component of a force along a direction is $F \cos \theta$; the useful part of an alternating current is the power factor $\cos \varphi$; the projection of any vector onto any axis is a cosine. On the unit circle it is simply the x-coordinate, and the identity $\sin^2 \theta + \cos^2 \theta = 1$ is the Pythagorean theorem on a triangle of hypotenuse 1.

Where it goes wrong. The most frequent error is picking the wrong leg: the adjacent side is the one touching the angle that is *not* the hypotenuse, and since the hypotenuse also touches the angle, that phrasing is exactly where people slip. The second is degree-versus-radian mode. This solver handles the conversion, but a phone calculator left in radians returns $\cos(35) = -0.903$, a negative number where a positive one belongs, which is at least loud enough to notice. The third is a domain limit that is really a geometry lesson: a/h can never exceed 1, because a leg cannot outrun the hypotenuse, so an arccos that refuses to evaluate is telling you the two lengths do not form a right triangle. Finally, solving for the hypotenuse divides by $\cos \theta$, which collapses toward zero as the angle nears 90° : near-vertical geometry makes this rearrangement extremely sensitive to a small error in the angle.

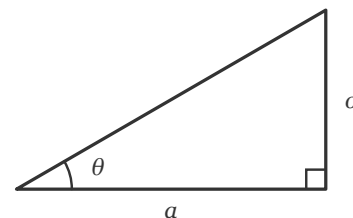


Figure 4.3: right-triangle tangent ratio (toa).

$\sin \theta = o/h$ by $\cos \theta = a/h$ cancels the hypotenuse and leaves o/a , which is the formula on this page and the reason the hypotenuse drops out of the problem.

Here is the fact that makes this the most quietly useful ratio of the three: **tangent is slope**. Rise over run is opposite over adjacent, so $m = \tan \theta$, and every grade, pitch and

gradient you have ever seen is a tangent in disguise. A 6% road grade is $\arctan(0.06) = 3.43^\circ$. A 6:12 roof pitch is $\arctan(0.5) = 26.6^\circ$. A 1:12 ramp is 4.76° . If you have ever wondered why a highway sign warning of an 8% grade describes something that looks nowhere near eight degrees, this is why, and it is worth noticing that a 100% grade is 45° , not vertical.

The classic use is an angle of elevation. Stand 50 m out from a tower on level ground and sight the top at 31° : the rise above your eye is $50 \tan 31^\circ = 30.0$ m. Backwards, $\theta = \arctan(o/a)$, and the principal branch is exactly right here: two positive legs always land in 0° – 90° , so there is no ambiguity to resolve.

Four cautions, in the order they cost people money. First, that tower is not 30 m tall. The formula returns the rise above the *instrument*, so you must add your eye height, or

the tripod height, to get the real total: a systematic error of a metre and a half that stays invisible because the answer looks reasonable. Second, a is the *horizontal* distance. On sloping ground a taped distance runs along the slope and is longer than the run, which inflates the result; that is why survey instruments reduce slope distance to horizontal before anything else happens. Third, tangent has no ceiling, it runs to infinity at 90° , and grows viciously sensitive as it approaches. At 30° a half-degree error in the sighting shifts the height by about 2%; at 80° the same half degree shifts it by 5%, and at 85° by 10%. Standing farther back and sighting at a shallower angle is nearly always the more accurate measurement. Fourth, as ever, check whether your calculator is in degrees.

Worked example 4.3. Given $\theta = 30^\circ$, $a = 90$ m; find o (opposite side). *30° elevation, 90 m from base $\rightarrow$ 51.9615 m height ($30\sqrt{3}$).*

4.4 Pythagorean Theorem

(Math: Functions & Applications 11-31) $a^2 + b^2 = c^2$

Where,

a = Leg a (m)

b = Leg b (m)

c = Hypotenuse c (m)

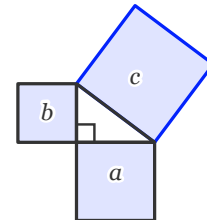


Figure 4.4: pythagorean theorem.

In a right triangle, $a^2 + b^2 = c^2$, where c is the hypotenuse: the side facing the right angle. Written out, it says that the square built on the long side has exactly the same area as the two squares built on the short sides put together, and that is worth pausing on, because it is a statement about **areas** that lets you calculate a **length**. That is the whole trick, and it is why the squares and the square root are there rather than something simpler.

There are hundreds of proofs; the one worth carrying around takes two copies of a square of side $a + b$. In the first, arrange four copies of the triangle in the corners so the

leftover space forms two squares, of areas a^2 and b^2 . In the second, slide the same four triangles into a pinwheel so the leftover space is a single tilted square of area c^2 . Same big square, same four triangles removed, so what remains must match. No algebra, and you can do it with paper. The converse is true as well, and it is the half that earns its living on site: if three measured lengths satisfy the relation, the angle between the two short ones *is* square. That is why a layout crew measures 3, 4 and 5 to set a corner rather than trusting a framing square.

Use the biggest triangle the space allows: a 3–4–5 in feet leaves a corner good to perhaps half a degree, while a 12–16–20 divides that error by four. A rafter run of 5.4 m with a rise of 2.2 m needs a length of $\sqrt{5.4^2 + 2.2^2} = 5.83$ m before the tail cut. Two triples are worth memorising because they come out whole: 3–4–5 and 5–12–13.

Three ways it goes wrong. The first is putting a leg where the hypotenuse belongs: c is always the longest side, so if the answer comes back shorter than something you typed, the sides are in the wrong slots. The second is dropping the squares: the legs 3 and 4 do not make a 7, they make a 5, and the shortcut across a rectangular lot saves far less than people expect. The third is applying it to a triangle that has no right angle at all; for those, the law of cosines carries a correction term and reduces to this the moment the angle reaches 90° . Solving for a leg has a built-in honesty check: $a = \sqrt{c^2 - b^2}$ needs $c > b$, and if it is not, the square root turns imaginary because the triangle you described cannot be drawn.

Worked example 4.4. Given $a = 3$ m, $b = 4$ m; find c (hypotenuse c). *legs 3 m and 4 m $\rightarrow$ hypotenuse 5 m.*

4.5 Law of Sines

(Math: Functions & Applications 11-32)
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

Where,

a = Side a (m)

A = Angle A (opposite side a) ($^\circ$)

b = Side b (m)

B = Angle B (opposite side b) ($^\circ$)

The law of sines says every triangle keeps a fixed exchange rate between sides and the sines of their opposite angles: bigger angle, proportionally bigger opposite side. (The common ratio is in fact the diameter of the triangle's circumscribed circle.) It is the tool of choice when you know an angle–side opposite pair: navigators and surveyors have leaned on it for centuries, since two sighted angles and one measured baseline pin down every other distance by

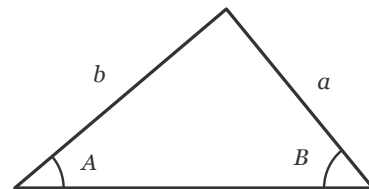


Figure 4.5: law of sines.

triangulation. Example: a baseline $b = 100$ m with angles $A = 40^\circ$ and $B = 65^\circ$ gives $a = 100 \times \sin 40^\circ / \sin 65^\circ \approx 70.9$ m, no tape measure across the river required.

The solver handles this law for the sides only. Solving for an angle would require arcsin, and the principal branch cannot tell an acute angle from its obtuse supplement: the classic *ambiguous SSA case*, where two different triangles fit the same data. The side rearrangements are single-valued closed forms with no such trap.

Worked example 4.5. Given $a = 10$ m, $A = 30^\circ$, $B = 45^\circ$; find b (side b). $a = 10$ m, $A = 30^\circ$, $B = 45^\circ \rightarrow b = 10\sqrt{2} = 14.1421$ m.

4.6 Law of Cosines

(Math: Functions & Applications 11-33) $c^2 = a^2 + b^2 - 2ab \cos C$

Where,

a = Side a (m)

b = Side b (m)

c = Side c (opposite angle C) (m)

C = Angle C (between a and b) ($^\circ$)

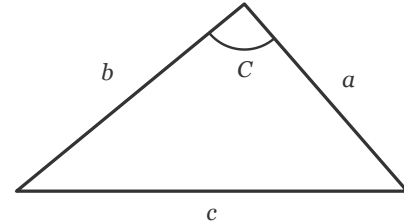


Figure 4.6: law of cosines.

The law of cosines is the Pythagorean theorem with a correction term: when angle C is exactly 90° , $\cos C$ vanishes and $c^2 = a^2 + b^2$ reappears. Open the angle wider and the correction adds length; squeeze it and the correction subtracts. Euclid proved both cases geometrically around 300 BCE, and the Persian astronomer al-Kashi gave the modern trigonometric form in the 1400s: in France the result still carries his name. A surveyor's example: from one

station, two landmarks lie 8 km and 5 km away with 60° between the sightlines, so their separation is $\sqrt{64 + 25 - 2 \cdot 8 \cdot 5 \cdot \cos 60^\circ} = \sqrt{49} = 7$ km.

Solved for the angle, $C = \arccos((a^2 + b^2 - c^2)/(2ab))$ uses the principal branch of arccos, spanning 0° to 180° : precisely the range of a triangle's interior angle, so the answer is unique with no ambiguous case. The three sides must obey the triangle inequality, or the arccos argument escapes $[-1, 1]$ and no triangle exists.

Worked example 4.6. Given $a = 5$ m, $b = 8$ m, $C = 60^\circ$; find c (side c (opposite angle c)). Sides 5 m and 8 m at $60^\circ \rightarrow$ third side 7 m.

4.7 Triangle Area (Two Sides and Included Angle)

(Math: Functions & Applications 11-34) $A = \frac{1}{2} ab \sin C$

Where,

A = Triangle area (m^2)

a = Side a (m)

b = Side b (m)

C = Included angle between a and b ($^\circ$)

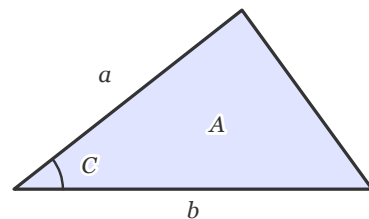


Figure 4.7: triangle area (two sides and included angle).

This is half base times height with the height supplied by trigonometry rather than by a tape. Take side a as the base. Drop a perpendicular from the far end of side b down to it; that perpendicular is the opposite leg of a right triangle whose hypotenuse is b and whose angle is C , so its length is exactly $b \sin C$. Substitute into $A = \frac{1}{2}(\text{base})(\text{height})$ and

you have $A = \frac{1}{2} ab \sin C$. Nothing new has been introduced: the formula's whole contribution is letting you skip a measurement that is awkward to take, because the height of a triangle usually hangs in mid-air.

Surveyors and land agents use it constantly. Two fence lines running 30 m and 42 m from a common corner, with 75° between them, enclose $\frac{1}{2}(30)(42)\sin 75^\circ = 608 \text{ m}^2$. The same arithmetic sizes a triangular section of roof, a gusset plate, or a wedge-shaped bay in a floor plan, and the solver runs it backwards to give the side a required area demands.

Two connections are worth carrying. Set $C = 90^\circ$, where $\sin C = 1$, and the expression collapses to $\frac{1}{2}ab$: the two sides are the legs of a right triangle. Since the sine can never exceed 1, that is also the *largest* triangle two given sides can make: for fixed a and b , the area is greatest when they meet squarely, and falls away toward zero as the angle closes to 0° or opens toward 180° . The second connection is to vectors: $ab\sin C$ is the magnitude of the cross product of two edge vectors, so this formula is the cross product wearing a triangle costume, and it is what underlies the shoelace method for polygon areas.

The failure mode here is a quiet one, which makes it worse. **C must be the angle enclosed between a and b** , not simply an angle of the triangle you happen to know. Feed in a

Worked example 4.7. Given $a = 7 \text{ m}$, $b = 8 \text{ m}$, $C = 30^\circ$; find A (triangle area). *Sides 7 m and 8 m with 30° between $\rightarrow$ area 14 m^2 .*

different angle and the arithmetic proceeds without complaint and returns a number that is wrong: there is no domain error to catch you, because nothing invalid happened. Check that the two sides you entered both radiate from the corner whose angle you entered. There is also a limit built into the geometry: the solver will not solve for C , because $\sin C = \sin(180^\circ - C)$, so a 40° corner and a 140° corner between the same two sides give identical areas and the area cannot tell them apart. And if what you actually hold is three side lengths rather than two sides and an angle, this is the wrong page: Heron's formula is the one that takes three sides.

Module 4 practice

41. Choosing the ratio: A grain conveyor is 16 m long along its slope and meets the level yard at 30° . *Calculate the height its upper end reaches.*

42. Choosing the ratio: A shoring brace meets the level floor at 60° , and its foot is set 4 m out from the wall it holds. *Determine the length of the brace.*

43. Heights and shadows: Standing 25 m from the foot of a stadium light mast, a student measures the angle of elevation to the lamp head as 60° . *Determine the height of the mast.*

44. Heights and shadows: Standing 90 m from the foot of a stadium light mast, a student measures the angle of elevation to the lamp head as 30° . *Determine the height of the mast.*

45. The sine law: On a bridge truss with no square corner, the 25 m member sits opposite a 30° joint, and a second member measures 30 m. Every joint on the drawing is marked acute. *Determine the angle opposite the 30 m member.*

46. The sine law: On a bridge truss with no square corner, the 20 m member sits opposite a 30° joint, and a second member measures 28 m. Every joint on the drawing is marked acute. *Determine the angle opposite the 28 m member.*

47. The cosine law: Two boundary lines leave the same corner post: one runs 6 m, the other 10 m, and the angle between them measures 120° . *Determine the distance between their far ends.*

48. The cosine law: A triangular parcel has boundaries of 7 m, 15 m and 13 m, with no square corner anywhere on it. *Determine the angle between the 7 m and 15 m boundaries.*

49. Area without the height: A surveyor records two boundaries of a triangular lot as 13 m and 12 m, with 30° between them at the corner post. *Calculate the area of the lot.*

50. Area without the height: A sail is cut as a triangle: two edges measure 5 m and 8 m, and the angle between them is 30° . *Determine the area of cloth in the sail.*

51. The Survey: Last job of the season. Lot ABCD is surveyed as two triangles across the diagonal BD. From corner A the boundary AB runs 60 m and the boundary AD runs 160 m, and the interior angle at A measures 60° . Across the diagonal, triangle BCD carries a 30° corner at C — the one facing the diagonal — and a square corner at B, facing the boundary CD. (The paper's table: $\cos 60^\circ = 0.50$ · $\sin 60^\circ = 0.87$ · $\sin 30^\circ = 0.50$ · $\sin 90^\circ = 1.00$.) Work each line down the page — every answer feeds the next. *Determine the total area of lot ABCD, one line at a time.*

52. The Survey: Bonus mark, worked backwards. A second parcel closes on three measured boundaries: 100 m, 160 m and 140 m. No angle was recorded in the field. (The paper's table: $\cos 60^\circ = 0.50$ · $\cos 90^\circ = 0.00$ · $\cos 120^\circ = -0.50$.) *Determine the angle between the 100 m and 160 m boundaries.*

MODULE 5

Sequences & Financial Math

Spotting the pattern · Arithmetic sequences · Arithmetic series · Geometric sequences · Geometric series · Simple interest · Compound interest · Annuities and loans · The First Paycheque

5.1 Arithmetic Sequence nth Term

(Math: Functions & Applications 11-35) $a_n = a_1 + (n - 1)d$

Where,

a_n = nth term

a_1 = First term

n = Term number

d = Common difference

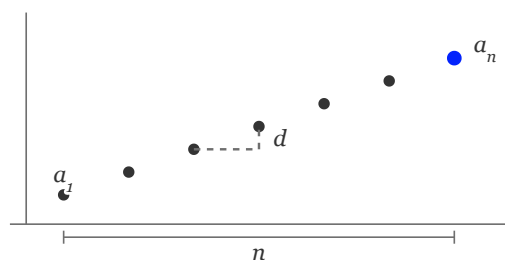


Figure 5.1: arithmetic sequence nth term.

An arithmetic sequence climbs (or falls) by the same amount every step, so the n th term is just the first term plus $(n - 1)$ copies of the common difference d . Theater rows work this way: if row 1 has 20 seats and each row adds 3 more, row 12 has $20 + 11 \times 3 = 53$ seats.

Run it backward to locate a value: with $a_1 = 5$ and $d = 4$, the term equal to 45 sits at $n = (45 - 5)/4 + 1 = 11$. If that comes out fractional, the value simply never appears in the sequence.

Worked example 5.1. Given $a_1 = 2$, $n = 10$, $d = 3$; find a_n (nth term). 2, 5, 8, ... 10th term $\rightarrow 29$.

5.2 Geometric Sequence nth Term

(Math: Functions & Applications 11-36) $a_n = a_1 r^{n-1}$

Where,

a_n = nth term

a_1 = First term

r = Common ratio

n = Term number

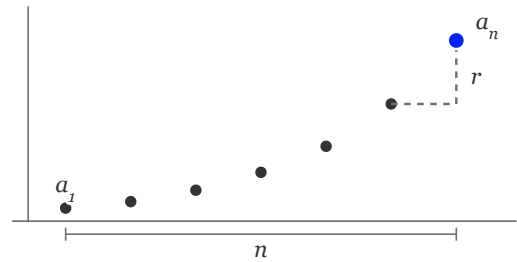


Figure 5.2: geometric sequence nth term.

A geometric sequence multiplies by the same ratio r every step, so the n th term is the first term times r raised to $(n - 1)$. Cell division is the classic case: starting from 1 cell that doubles each generation ($r = 2$), generation 11 has $1 \times 2^{10} = 1024$ cells. Ratios below 1 shrink instead: a ball rebounding to 60% of each bounce height that starts at 2 m reaches $2 \times 0.6^4 \approx 0.26$ m on the fifth bounce.

Solving for r extracts the per-step multiplier from two snapshots, and solving for n (with logarithms) answers "which step reaches this value?" This calculator keeps r positive so fractional positions stay real.

Worked example 5.2. Given $a_1 = 3$, $r = 2$, $n = 8$; find a_n (nth term). *3, 6, 12, ... 8th term $\rightarrow 384$.*

5.3 Arithmetic Series Sum

(Math: Functions & Applications 11-37) $S_n = \frac{n}{2} (2a_1 + (n - 1)d)$

Where,

S_n = Sum of the first n terms

n = Number of terms

a_1 = First term

d = Common difference

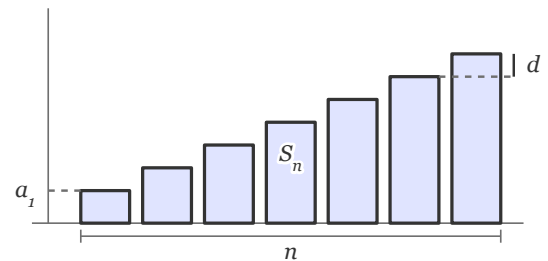


Figure 5.3: arithmetic series sum.

Write the series out, then write it out again underneath in reverse order, and add the two lines column by column. Every column comes to the same thing, the first term plus the last, because whatever one line gains going right, the other loses. There are n columns and you have counted the series twice, so $2S = n(a_1 + a_n)$, which is $S_n = \frac{n}{2}(a_1 + a_n)$. Unpack the last term as $a_1 + (n - 1)d$ and you have the form on this page. The trick is credited to Gauss at about age nine, told to add the whole numbers up to 100 and answering 5050 almost immediately, and whether or not the story is true it is the right way to see the formula.

Read the result plainly and it says: the sum is the number of terms times the *average* term, and for an arithmetic sequence the average is simply the midpoint between the first and the last. No other kind of sequence lets you get away with that.

Stacking problems fall out directly. A pile of pipe with 15 on the bottom row and one fewer per row for 10 rows holds $\frac{10}{2}(2 \times 15 + 9 \times (-1)) = 105$ lengths. An amphitheatre with 20 seats in the front row and three more in each row behind it seats $\frac{12}{2}(40 + 11 \times 3) = 438$ people over twelve rows. The solver also runs backwards, recovering the first term or the step from a known total.

The mistake that dominates this topic is the $(n - 1)$. Twelve rows have only eleven gaps between them, so the twelfth row is eleven steps up from the first, not twelve. It is the same off-by-one that makes a 100 m fence with posts every 10 m need eleven posts, and it survives into professional work because the wrong answer is close enough to look right. A related version: summing from the 5th term to the 20th covers 16 terms, not 15, because both ends count. Two smaller traps. The step d is signed: the pipe stack above needs $d = -1$, and entering 1 describes a pile that widens as it rises. And n has to be a whole positive count of terms; a fractional answer when solving backwards means no such series exists. Finally, notice the family resemblance to the trapezoid: average of the two ends times the extent. An arithmetic series is a trapezoid counted in discrete steps, which is a fair way to remember either formula from the other.

Worked example 5.3. Given $n = 10$, $a_1 = 2$, $d = 3$; find S_n (sum of the first n terms). $2 + 5 + 8 + \dots$ (10 terms)
 $\rightarrow S = 155$.

5.4 Geometric Series Sum

(Math: Functions & Applications 11-38)
$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

Where,

S_n = Sum of the first n terms

a_1 = First term

r = Common ratio

n = Number of terms

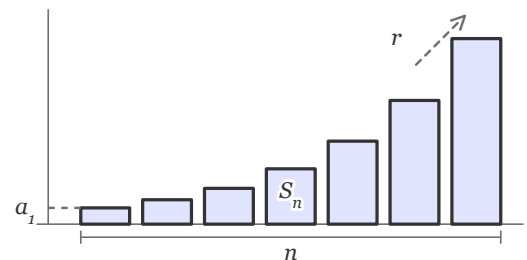


Figure 5.4: geometric series sum.

Adding the first n terms of a geometric sequence collapses to a single fraction: $a_1(1 - r^n)/(1 - r)$. The legendary chessboard reward (1 grain of wheat on the first square, doubling each square) sums to $(2^{64} - 1) \approx 1.8 \times 10^{19}$ grains over 64 squares, more wheat than the world has ever grown.

Shrinking ratios are just as practical: saving \$100 in month one but only 80% as much each following month accumulates $S = 100(1 - 0.8^{12})/(1 - 0.8) \approx \$465.64$ over a year, already close to the \$500 ceiling the infinite series would give.

Worked example 5.4. Given $a_1 = 3$, $r = 2$, $n = 8$; find S_n (sum of the first n terms). $3 + 6 + \dots + 384$ (8 terms)
 $\rightarrow S = 765$.

5.5 Simple Interest

(Math: Functions & Applications 11-39) $I = Prt$

Where,

I = Interest earned

P = Principal

r = Interest rate per period (decimal)

t = Number of periods

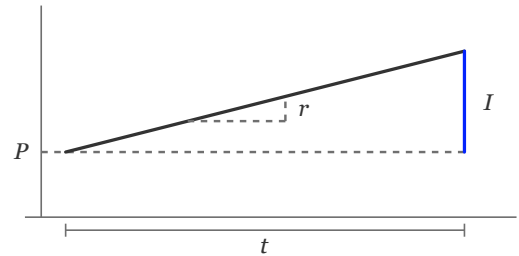


Figure 5.5: simple interest.

Simple interest pays only on the original principal, never on past interest: a \$2,000 loan at 6% per year for 3 years costs $I = 2000 \times 0.06 \times 3 = \$360$, exactly \$120 each year. Enter the rate as a decimal (6% $\rightarrow$ 0.06) and keep the rate and the time in the same period: a monthly rate with months, a yearly rate with years.

Short-term personal loans, car loans, and bonds' coupon payments often work this way. Compare with compound interest, where each period's interest joins the principal and the total pulls ahead of the simple-interest straight line.

Worked example 5.5. Given $P = 1000$, $r = 5\%$, $t = 3$; find I (interest earned). $\$1000$ at 5% for 3 periods $\rightarrow I = 150$.

5.6 Compound Interest (Periodic)

(Math: Functions & Applications 11-40) $A = P \left(1 + \frac{r}{n}\right)^{nt}$

Where,

A = Final amount

P = Principal

r = Interest rate per period (decimal)

n = Compounds per period

t = Number of periods

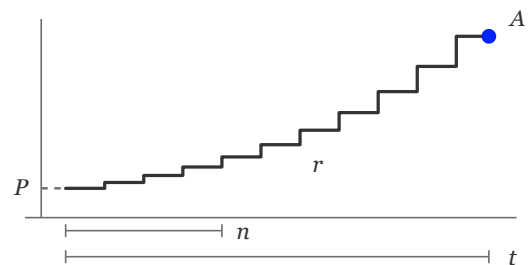


Figure 5.6: compound interest (periodic).

With periodic compounding, each period's rate r is split into n slices and applied n times, so past interest starts earning interest of its own. \$5,000 at 4% per year compounded monthly ($n = 12$) for 10 years grows to $A = 5000 \times (1 + 0.04/12)^{120} \approx \$7,454.16$: about \$50 more than yearly compounding would give, because 120 small boosts beat 10 large ones.

Solving for r recovers the rate a savings product actually paid between two statements, and solving for t answers "how long until my balance reaches A ?" Enter r as a decimal (4% $\rightarrow$ 0.04), and note that pushing n toward infinity lands on the continuous-compounding formula $A = Pe^{rt}$.

Worked example 5.6. Given $P = 1000$, $r = 6\%$, $n = 12$, $t = 10$; find A (final amount). $\$1000$ at 6%/yr monthly for 10 yr $\rightarrow A = 1819.40$.

5.7 Present Value

(Math: Functions & Applications 11-41)
$$PV = \frac{FV}{(1 + r)^t}$$

Where,

PV = Present value

FV = Future value

r = Discount rate per period (decimal)

t = Number of periods

Money later is worth less than money now, because money now could be invested. Present value runs compound growth in reverse: \$10,000 arriving in 8 years, discounted at 5% per year, is worth $PV = 10000 / 1.05^8 \approx \$6,768.39$ today. Enter the rate as a decimal (5% $\rightarrow$ 0.05).

Worked example 5.7. Given $FV = 10000$, $r = 5\%$, $t = 8$; find PV (present value). $\$10,000$ in 8 yr at 5% $\rightarrow PV = 6768.39$.

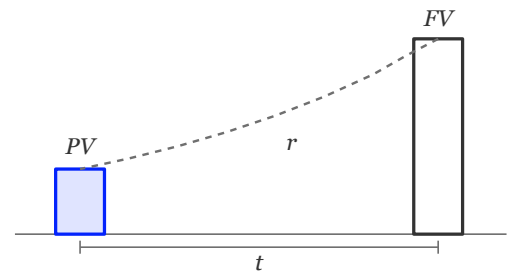


Figure 5.7: present value.

This one discount is the atom of finance: bond prices, mortgage balances, and a company's valuation are all sums of future cash flows each pulled back to today. Solving for r asks "what return does this deal imply?", and solving for t asks how long a target takes at a given rate.

5.8 Future Value of an Annuity (Regular Deposits)

(Math: Functions & Applications 11-42)
$$FV = D \frac{(1 + i)^n - 1}{i}$$

Where,

FV = Future value (\$)

D = Deposit each period (\$)

i = Interest rate per period

n = Number of deposits

Put the same amount away at the end of every period and each deposit compounds for a different length of time. The first sits longest, the last earns nothing at all, and adding up that staircase gives $FV = D[(1 + i)^n - 1]/i$. Three annual deposits of \$1,000 at 5% grow to $1000(1.1025) + 1000(1.05) + 1000 = 3152.50$, which the closed form reproduces exactly.

Read backwards, the same relation is the sinking fund: the deposit you need now to have a known sum later. A contractor who knows a \$60,000 truck needs replacing in

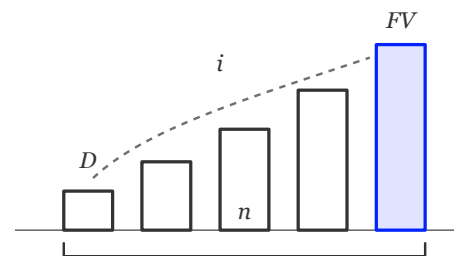


Figure 5.8: future value of an annuity (regular deposits).

seven years can ask what monthly transfer gets there, rather than discovering the answer when the old one dies. This is how equipment reserves, roof funds and condominium capital plans are actually built.

The version here assumes deposits at the *end* of each period, the ordinary annuity. If you deposit at the beginning instead, every dollar earns one extra period, and the whole answer is simply larger by a factor of $(1 + i)$.

Worked example 5.8. Given $D = 1000$ \$, $i = 5\%$, $n = 3$; find FV (future value). 3 deposits of \$1,000 at 5% $\rightarrow$ \$3,152.50.

5.9 Loan Payment (Amortized Loan or Mortgage)

(Math: Functions & Applications 11-43)
$$M = \frac{Pi}{1 - (1 + i)^{-n}}$$

Where,

M = Payment per period (\$)

P = Principal borrowed (\$)

i = Interest rate per period

n = Number of payments

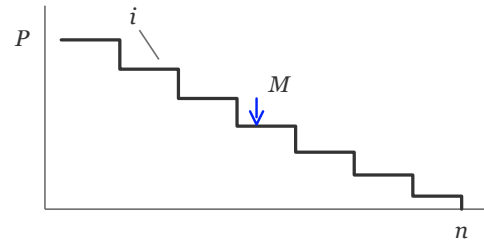


Figure 5.9: loan payment (amortized loan or mortgage).

Every amortized loan is a promise to hand over the same amount, on the same day, until the balance is gone. Working out that amount looks hard and is not: the payments form a geometric series, and summing it gives $M = Pi / (1 - (1 + i)^{-n})$. The classic case is a \$200,000 mortgage at 6% nominal over 30 years, which means $i = 0.06/12 = 0.005$ per month and $n = 360$ payments, giving \$1,199.10.

The single most common mistake is feeding it an annual rate alongside a monthly count. The rate and the payment count must describe the *same* period, always. A second trap is national: Canadian mortgages are compounded semi-annually by law, so a 6% Canadian mortgage has a monthly

rate of $1.03^{1/6} - 1 \approx 0.004939$, not 0.005, and the payment comes out slightly lower than the American figure above.

A quick sanity check lives inside the formula. Set $n = 1$ and it collapses to $M = P(1 + i)$, which is exactly right: one payment repays the principal plus a single period's interest. Note also which variable is missing here. You can solve for the principal, and you can solve for the number of payments by taking logs, but there is no closed-form solution for the interest rate. Recovering i from a payment means finding the root of a degree- n polynomial, which is why every spreadsheet computes it by iteration.

Worked example 5.9. Given $P = 200000$ \$, $i = 0.005$, $n = 360$; find M (payment per period). \$200,000 at 6% over 30 years $\rightarrow$ \$1,199.10 per month.

Module 5 practice

53. Arithmetic sequences: The 14th term of an arithmetic sequence is 57, and the common difference is 4. *Determine the first term.*

54. Arithmetic sequences: The 15th term of an arithmetic sequence is 64, and the common difference is 4. *Determine the first term.*

55. Arithmetic series: A theatre has 20 rows. The front row holds 14 seats, and each row behind holds 3 more than the row in front of it. *Calculate the total number of seats in the theatre.*

56. Arithmetic series: Pipes are stacked in 10 rows: 19 pipes on the bottom row, and each row above holds one fewer than the row below. *Determine how many pipes are in the stack.*

57. Geometric sequences: The 9th term of a geometric sequence is 512, and the common ratio is 2. *Determine the first term.*

58. Geometric sequences: The 5th term of a geometric sequence is 405, and the common ratio is 3. *Determine the first term.*

59. Geometric series: A message is shared: 5 students see it in the first hour, and each hour after that 2 times as many new students see it as in the hour before. This runs for 7 hours. *Calculate the total number of students who have seen the message.*

60. Geometric series: A message is shared: 4 students see it in the first hour, and each hour after that 2 times as many new students see it as in the hour before. This runs for 7 hours. *Calculate the total number of students who have seen the message.*

61. Simple interest: A student lends \$500 to a cousin at 5% simple interest per year, to be repaid after 4 years. *Calculate the interest owed.*

62. Simple interest: A student lends \$1,000 to a cousin at 5% simple interest per year, to be repaid after 2 years. *Calculate the interest owed.*

63. Compound interest: A savings bond pays out \$1,210 in 2 years. Money in this plan grows at 10% per year, compounded annually. *Determine what the payout is worth today.*

64. Compound interest: A savings bond pays out \$1,210 in 2 years. Money in this plan grows at 10% per year, compounded annually. *Determine what the payout is worth today.*

65. Annuities and loans: A student puts \$2,000 into a savings plan at the end of every year for 5 years. The plan earns 5% per year, compounded annually. *Determine the plan's value just after the final deposit.*

66. Annuities and loans: A family signs a mortgage of \$150,000 at 6% per year, compounded monthly, amortized over 30 years — 360 equal monthly payments. *Calculate the monthly payment.*

67. The First Paycheque: First job, first paycheque — and three places to put the signing bonus. Plan A: \$1,800 into a simple-interest account at 5% per year. Plan B: the same \$1,800 into a GIC at 10% per year, compounded annually. Plan C: half now, half later — \$900 at the end of each year into that same 10% account. Two years. Work each line — every answer feeds the next. *Determine each plan's balance after the two years, one line at a time — then crown the winner.*

Answer key

1. -3 (no unit)	24. 6 hours	47. 14 m
2. 3 (no unit)	25. 8 (no unit)	48. 60 °
3. 9 (no unit)	26. 4 (no unit)	49. 39 m ²
4. 4 (no unit)	27. 4 doublings	50. 10 m ²
5. 3 (no unit)	28. 3,200 cells	51. 140 m
6. 6 (no unit)	29. 0.5 Hz	52. 60 °
7. -39 (no unit)	30. 0.4 s	53. 5 (no unit)
8. 0 (no unit)	31. 4.5 m/s	54. 8 (no unit)
9. 6 (no unit)	32. 5 m	55. 850 seats
10. 5.5 (no unit)	33. 1 s	56. 145 pipes
11. 11.1 m	34. 3.5 s	57. 2 (no unit)
12. 25.6 m	35. 18.8 m/s	58. 5 (no unit)
13. 2 s	36. 3.1 m/s	59. 635 students
14. 10 m/s	37. 8 cm	60. 508 students
15. 25 %	38. 12.6 s	61. 100 \$
16. -20 %	39. 2 Hz	62. 100 \$
17. 2,500 beetles	40. 400 Hz	63. 1,000 \$
18. 25 %	41. 8 m	64. 1,000 \$
19. 20 %	42. 8 m	65. 11,051.3 \$
20. 20 %	43. 43.3013 m	66. 899.33 \$
21. 30 mg	44. 51.9615 m	67. 1,980 \$
22. 15 minutes	45. 36.8699 °	
23. 6,400 cells	46. 44.427 °	